PI

THE ELEMENTS OF MECHANICS,

COMPREHENDING

STATICS AND DYNAMICS.

WITH

A COPIOUS COLLECTION OF

MECHANICAL PROBLEMS.

INTENDED FOR THE USE OF

MATHEMATICAL STUDENTS IN SCHOOLS AND UNIVERSITIES.

WITH NUMEROUS PLATES.

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PREFACE.

The following Treatise is an attempt to exhibit, in small compass, the principles of Mechanical Science in its present improved state, and to supply the English student with a clear and comprehensive manual of instruction on this important branch of Natural Philosophy.

Our language already possesses some very valuable works in this department of science, as for instance, the treatises of Professors Gregory and Whewell; works which, for the abundance of real information that they convey, are not, perhaps, to be equalled by any similar performances of our continental neighbours.

The bulk and consequent high price, however, of these works must necessarily place them beyond the reach of many students desirous to be informed on the subjects of which they treat; and there can, I think, be no doubt that at a time like the present, when a taste for analytical science is so widely extending itself, a treatise, of moderate price, on Analytical Mechanics, if well executed, would prove acceptable both to teachers and to students.

Under these impressions I have been led to undertake this elementary Treatise, with the hope that by economizing the paper, and adopting a small clear type, I might be able to compress into one small volume a course of instruction on elementary mechanics, of extent amply sufficient for all the purposes of academical education. My desire, however, having been to teach the elements of the science, not to write a book, I have spared no pains to render the whole clear and intelligible; to develope the several theories with as much simplicity as I could; to explain fully the meaning and extent of the various analytical expressions in which these theories are embodied; and finally, to illustrate each by a sufficient number of useful and interesting practical examples.

To what extent these endeavours may have been successful, it is for others now to determine; but, from the very flattering reception which my former mathematical publications have met with, both in this country and in America, I am encou-

raged to hope that the present volume will be found, upon examination, not altogether undeserving of notice.

The work is divided into two principal divisions, STATICS or the theory of Equilibrium, and DYNAMICS or the theory of Motion; and these are again subdivided into sections and chapters. A very short account of these will suffice here, as a copious analysis is presented in the table of contents.

The first section of the Statics treats on the equilibrium of a point, viewed under two aspects: first, as entirely free; and second, when constrained to rest on a given curve or surface. Into this section too is introduced the theory of the funicular polygon and catenary: this, I am aware, is not in strict accordance with a scientific arrangement of the parts of the subject, nor do I consider such arrangement to be absolutely essential in an elementary treatise; the principle which with me has all along governed the arrangement is this, viz., so to dispose the several topics that each may present itself to the student precisely at that place where he is best prepared to receive it, and thus the acquisition of the whole be facilitated.

The second section is on the theory of the equilibrium of a rigid body delivered in all its generality, and applied to a variety of examples. The closing chapter of this section is devoted to a subject of considerable practical importance, *the strength and stress of beams;* for the principal materials of it I am indebted to Mr. Barlow's experimental inquiries on this subject.

These two sections comprise the theory of Statics; the second part, or Dynamics, is divided into three sections; the first being on the rectilinear motion of a free point, the second on its curvilinear motion, and the third discussing the general theory of the motion of a solid body.

In the opening chapter of the first section, the fundamental equations of motion are deduced from simple and obvious considerations, and pains are taken to give a clear and distinct meaning to the several analytical expressions involved in these fundamental equations. All these are then fully illustrated by interesting practical exercises.

In the second section a pretty comprehensive view is taken of the theory of curvilinear motion; and some attempts have been made to simplify those parts of it which seemed most to require simplification.

The last section, on the motion of a solid body, is the most

extensive, as well as the most difficult, and will be found to embrace a great variety of important particulars, treated, I hope, with sufficient clearness to be abundantly intelligible to an ordinary student; several interesting dynamical problems are interspersed throughout this section, and to the end is appended a miscellaneous collection, as further illustrative of the use and application of the general theories before established, especially of the *Principle of D'Alembert*, on account of the importance of this principle in a great variety of dynamical inquiries.

For the manner in which the subjects here briefly enumerated are discussed, I must now refer to the book itself; and shall be glad if it be thought calculated to promote, in any degree, the study of the science, or to form a useful introduction to works of higher pretensions and of acknowledged ability.

J. R. YOUNG.

April 10, 1832.

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PART I.

ELEMENTS OF STATICS.

INTRODUCTION.

Article (1.) MECHANICS, taken in its most extensive acceptation, is the science which embraces all inquiries respecting the equilibrium and motion of bodies, whether solid or fluid, and, therefore, constitutes a very large and important part of NATURAL PHILOSOPHY, or that vast body of knowledge which explains the laws that govern the various operations of nature. It is usual however to give a more limited signification to the term Mechanics, and to treat under that denomination only of the equilibrium and motion of *solid* bodies. The theory of the equilibrium of solid bodies is called STATICS; the theory of their motion DYNAMICS; these, therefore, are the two great branches of the science of Mechanics.

(2.) If a body be submitted to any influence which would, if not opposed by an equal counteracting influence, move it, such influence, whatever it be, is called *force*. The term force, therefore, as employed in Mechanics, applies not merely to what, in common language, we understand by *power* or physical energy, but also to every cause which either produces or tends to produce motion, however hidden or inexplicable that cause may be.

(3.) A body subjected to the action of a force or moving influence, ought, necessarily, to move in the direction of that force; towards it, if the force draw or attract it, and from it, if the force push or repel it. Hence, it is the tendency of a body influenced by a single force applied to it at rest to move in a straight direction. But if to the same point two equal and directly opposite forces are applied, the tendency to motion in one direction being equal to the tendency in the opposite direction, the point will necessarily remain at rest, and will be as much prepared to obey the influence of any third force as it was before the two counterbalancing forces, of which we have just spoken, were applied.

It is obvious, that although two equal forces be applied to the same point they will not keep that point at rest, unless they are directly opposite as well as equal. This indeed may be easily proved thus: Suppose two equal forces P and Q (fig. 1,) tend to draw the point M in their respective directions MP, MQ, which are not opposite to each other; if we suppose the point to remain at rest, let us

introduce a third force P′ equal and opposite to P, that is, tending to draw M towards P′ with the same energy as P tends to draw it towards P. Now the point being acted upon by three forces, of which two, viz. P and P′, are in equilibrium, the point will tend to move in the direction MQ of the third force Q. But, by hypothesis, the two forces P, Q are in equilibrium; hence the tendency to motion must be in the direction MP′ of the third, which is absurd.

(4.) As it is the business of Statics to investigate the laws of forces in equilibrium, it will readily occur to the student that one of the principal problems of this branch of Mechanics is to determine, from knowing the magnitudes and directions of forces applied to a point, what must be the magnitude and direction of that counterbalancing force, which would prevent motion ensuing.

We may estimate the magnitudes of the forces of which we speak by means of weights; for it is plain that whatever influence at P (fig. 2,) solicits M, M may yet be kept in its place by the counteraction of some weight W tending to draw M in the opposite direction MP′. If for the force P were substituted another, which, in order to keep M unmoved, would render it necessary to double the weight W, we should then say that this new force was double the former, and, in like manner, the ratio of any two forces acting separately at P would be determined by the ratio of the separate counteracting weights acting in the directing MP′. We see that weight is a very suitable representative or measure of force; but for all the purposes of comparison it matters not by what we represent a force, taking care only that the representing quantities shall have the same ratio to one another as the forces represented, we are, therefore, at liberty to choose that mode of representation most conducive to the ends in view, viz. the investigation of the theory of equilibrating forces. We accordingly represent a force by a line drawn from the point on which it acts, called the point of application, in the direction of the force. The length of this line for one of the forces of the equilibrating system may be arbitrary, but for any other of the forces the length of the representing line will be to the former as the represented force is to the former force. Hence the theory of statics is reduced to that of lines and angles.

(5.) We have just said, that a force is represented by a line drawn from the point of application in the direction of that force; but we are at liberty to consider any point of this direction as the point of application of the force, and not merely the material point on which it acts: thus it matters not whether the force acting upon M (fig. 2,) to pull it in the direction MP be applied to the point P or to any other point in MP, provided we consider MP to be a perfectly inextensible line connecting P with M. Or if P be a repulsive force tending to push M in the direction MP′ then, supposing MP to be a perfectly

rigid line, it matters not whereabouts in this line P may be. All this is too obvious to require any laboured proof, and we shall now proceed to the general theory of equilibrating forces acting on a free point, viz., the point where the directions of these forces meet, and, to avoid confusion, we shall generally consider the forces concerned as *pulling* forces; for a pushing force may obviously be always supplied by a pulling force of the same intensity, and acting in an opposite direction.

SECTION I.

ON THE EQUILIBRIUM OF A POINT.

CHAPTER I.

ON THE COMPOSITION AND RESOLUTION OF CONCURRING FORCES.

Article (6.) By *concurring forces* we are to understand forces of which the directions all meet in a point, upon which point they simultaneously act; and they are said to be situated in the same plane when their directions are all in the same plane. To determine the *resultant* of two such forces, that is, a single force equally effective with the two, is a problem of great importance, and to this the present chapter will be chiefly devoted. The simplest case is that in which the concurrent forces act not only in the same plane, but even in the same straight line; if in this case the forces should both conspire or tend to move the point in the same direction, then their resultant would be equal to their sum; for it is plain that the weight W (fig. 2,) has the same effect on M as two smaller weights, together equal to W. If the two forces are opposite, then their resultant must be equal to their difference, and the direction of it towards the greater force, because so much of the greater force as is equal to the less force which opposes it, is employed in keeping the point in equilibrium, and the tendency to motion is the effect of the remaining force.

If, instead of two conspiring forces, there were three, then, by adding together two, we should obtain the resultant of those two, and this, added to the third, would furnish the resultant of the three; and in like manner the resultant of four, or of any number of conspiring forces, is found by merely adding together the component conspiring forces. If there be two systems of conspiring forces directly opposed to each other, they may thus be reduced to a single pair of opposing forces, and the difference of these two will be the resultant of the whole system. The direction, as well as the magnitude of this resultant, will be expressed algebraically, if we agree to consider the forces which conspire in one direction as all *positive*, and those which conspire in the opposite direction, as all *negative*; for we may then say that *the resultant of any number of forces acting in the same straight line, is equal to the sum of those forces;*

the sign of this sum pointing out the direction. This theorem, of course, includes the case in which the forces all pull one way, that is, where there is but a single system of conspiring forces.

Having disposed of this simple case of the general problem, we are now to determine the magnitude and direction of the resultant of *any two* concurring forces situated in a plane; we speak of only two concurring forces, because, as we shall soon see here, as in the case just considered, that when we know how to compound two, the composition of any number can present no difficulty.

There are several ways of arriving at the solution of this important problem, but the simplest process with which we are acquainted is that given by *Professor Gregory*, in his valuable treatise on Mechanics, who conducts the investigation as follows:

(7.) *Prop.* The equivalent of several forces situated in one plane, is in the same plane.

For if we suppose the equivalent to be out of the plane of the forces, on either side, we may always find a line on the other side of the plane situated in a perfectly similar manner; and since there can be no reason why the resultant should be in one of these directions, rather than in the other; it is therefore in neither of them, unless we admit the absurd consequence that it is in both, that is, unless we admit that the same forces acting in like manner can produce two distinct effects.

Cor. The resultant of two equal forces must be in their plane; and it must be in the line which bisects the angle of their direction, since there is no reason why it should tend more to one side than to another.

(8.) *Prop.* If to a material point, already kept in equilibrio by a system of forces, another system is applied also in equilibrio, this will not destroy the pre-existing equilibrium: this is manifest.

Cor. Hence, if the three forces, C, C′, O, (fig. 3,) are in a state of equilibrium, and if each of the forces were doubled, or tripled, or quadrupled, &c., or if they were halved, quartered, &c., or changed in any proportion, the equilibrium would remain so long as they continued to act in the same directions CP, C′P, OP.

Cor. 2. Hence, also, since the resultant R is always equal and opposite to one of the forces (as O), it follows, that when the magnitudes of equilibriated forces concurring in a point are made to vary in any ratio, the resultant retains its position but changes its magnitude in the same ratio.

(9.) *Prop.* If three equal forces are inclined to one another in angles each of 120 degrees, any one of them will balance the joint action of the other two. This is likewise incontrovertible; for neither of the forces can prevail.

(10.) *Prop.* Two equal forces inclined in an angle of 120 degrees,

have for their equivalent a third, which has the direction and proportion of the diagonal of the rhombus constructed on the lines which represent the forces.

For if C, C′ are the forces (fig. 3,) acting on the point P, the force O whose measure is OP=CP=C′P, and is situated so that the angles CPO and C′PO are each equal to CPC′, will (9) ensure the equilibrium. But RP, the measure of the equivalent R, is equal and opposite to OP (6); therefore CP=PR=C′P and because angle CPR=60°=C′PR, CR=CP and C′R=C′P. Consequently CPC′R is a rhombus, and RP, the representative of the equivalent of the forces C, C′, is its diagonal.

Cor. If half the angle CPC′ be denoted by a, we shall have PD=PC cos. a=C cos. a, whence the equivalent RP=2 C cos. a.

(11.) *Prop.* Any two equal forces have for their equivalent the diagonal of the rhombus constructed on the right lines which represent them in magnitude and direction.

1. If this proposition be true with regard to any two equal forces C, C′ acting in the directions CP, C′P (fig. 4), and forming with their resultant R the angles CPR, C′PR each for example equal to a, it is true, likewise, for two other equal forces c, c' acting according to the directions cP, c'P, which bisect those angles. In this case c may be considered (7 Cor.) as the resultant of two equal forces, x and y, acting in the directions CP, RP; and in like manner, c' may be considered as the resultant of two other equal forces, x' and y', acting in the directions C′P, RP: so that, in lieu of the two equal forces c, c', we may consider four equal but unknown forces x, x', y, y' acting in the directions just assigned them. The two first of these, x, x', acting in the directions CP, C′P, have, by hypothesis, the diagonal of the rhombus for their resultant: that is, they are equivalent to a force expressed by 2 x cos. a acting in RP, therefore the resultant z of c and c' will be equal to $2y+2x$ cos. a; but $x=y$, therefore $z=2x$ (1+ cos. a). Now the angles CPR, c P c' being each equal to half CPC′, are equal to each other; and c being the resultant of two equal components acting in CP and RP, we have

$$z:c::c:x=\frac{c^2}{z}$$

Substituting this value of x for it in the preceding equation, we obtain

$$z=\frac{2c^2}{z}(1+\cos.\ a)\ \therefore\ z^2=2c^2(1+\cos.\ a)\ \therefore\ z=c\sqrt{2(1+\cos.\ a)}.$$

But it is known that cos. $\frac{1}{2}$ $a=\sqrt{\frac{1+\cos.\ a}{2}}$, (*Gregory's Trigonometry*, p. 46,) whence, by substitution, $z=2c$ cos. $\frac{1}{2}$ a. Consequently, the proposition if true for a is true for $\frac{1}{2}$ a.

2. In exactly the same manner may the proposition be proved true with respect to the half of $\frac{1}{2}a$ or $\frac{1}{4}a$, and in succession for $\frac{1}{8}a$, $\frac{1}{16}a$, $\frac{1}{32}a$, &c. That is, since it is true, (9) when the angle CPC′ is measured by $\frac{1}{3}$ of the circumference, it is likewise true when the angle between the equal components is measured by $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, &c. of the circumference, where the series may be continued *sine limite*.

3. If the proposition be demonstrated for the three angles a, b, and $a-b$, it will be true for the angle $a+b$; that is, if we take two equal components c and c', making with their resultant x, angles $=a+b$, we shall have $x=2c \cos. (a+b)$. Thus, if in fig. 5 the angles CPR, C′PR are each equal to a, and cPC, CPd, c'PC′, C′Pd', each equal to b: conceiving two forces dP, d'P each equal to c, their resultant will, by hypothesis, be $=2c \cos. (a-b)$, because $d\text{PR}=a-b$; and this quantity subtracted from the resultant of c, c', d, d', will give x. But c and d have their resultant C acting in CP, and $=2c \cos. b$; the same thing holding with respect to c' and d', we have two forces equal to C and equivalent to one, which is $2\text{C} \cos. a$ or $4c \cos. a \cos. b$; whence $x=4c \cos. a \cos. b -2c \cos. (a-b)$. But $\cos. a \cos. b=\frac{1}{2} \cos. (a+b)+\frac{1}{2} \cos. (a-b)$. (*See Gregory's Trig.*, p. 44, art. 20). Which value of $\cos. a \cos. b$, substituted for it in the preceding equation, gives $x= 2c \cos.(a+b)$. So that the proposition when true for a, b, and $a-b$, is true for $a+b$.

4. Let b be taken as small as we please in the series, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$ &c. and let a be the preceding term in the series, then a, b, $a-b$, $a+b$, are $2b$, b, b, and $3b$, respectively, in each of which the proposition holds: again, if $a=3b$, $a+b=4b$; if $a=4b$, $a+b=5b$, &c. So that the theorem is demonstrated for all angles in the series b, $2b$, $3b$, $4b$, $5b$, &c., in which b may be taken of a magnitude less than any one which can be assigned. Consequently, the theorem is true with respect to any rhombus whatever; for let any rhombus be proposed, which it is affirmed is an exception to this proof; we can, it is obvious, by choosing b lower than any assigned angle, and taking a suitable multiple of it, approach nearer the accepted angle than by any assignable difference, that is, we show that our theorem is applicable to the angle itself.

(12.) *Prop.* Any two forces having the ratio of the sides of a rectangle, and whose directions coincide with those sides, have for their equivalent the diagonal of that rectangle.

Let the two forces C, C′ (fig. 6,) act in the directions CP, CP′, which comprise the right angle P: complete the parallelogram CPC′R, and draw its diagonals; parallel to CC′, draw cc' terminated by Cc, C′c', which are drawn parallel to the resulting diagonal. Conceive c and c' to be two equal forces acting in the equal lines cP,

c'P, opposite to each other, and consequently annihilating each other's effects; then cPDC and c'PDC$'$ being rhombi, the force CP is the equivalent of cP, DP, and C$'$P that of c'P, DP, by the preceding proposition. Therefore, the components CP, CP are the same in effect as the opposite ones cP, c'P together with DP, DP; that is, the equivalent sought is 2DP or RP, the diagonal of the parallelogram.

Cor. Since RP : rad. :: CP : cos. CPR :: C$'$P : cos. C$'$PR, we have the resultant equal to either component divided by the cosine of the angle which it makes with the resultant.

(13.) *Prop.* Any two forces whatever have their equivalent expressed in magnitude and direction by the diagonal RP of the parallelogram constructed on the lines CP, C$'$P, which represent these forces.

Having completed the parallelogram CPC$'$R (fig. 7,) on the given sides, draw cc' perpendicular, and Cc, C$'c'$ parallel, to the diagonal; demit also CD, C$'$D$'$ perpendicular to the diagonal: then will Cc PD, C$'c'$ PD$'$ be rectangles, and the triangles CRD, C$'$PD$'$ equal in all respects, consequently Cc=DP, RD=D$'$P, and cP=c'P. The addition of the equal forces c, c', acting in the opposite directions cP, c'P, will make no difference in the state of the system; and since the components DP, cP, have CP for their resultant, and the components D$'$P, c'P, the resultant C$'$P, (by the preceding proposition,) we may, instead of the original forces CP, C$'$P, substitute the forces cP, c'P, DP, D$'$P, of which the two former destroy each other's effects, and the latter DP, D$'$P, are manifestly equal to RP; that is, the resultant of the two forces CP, C$'$P, is equal to the diagonal RP of the parallelogram.

"Thus have we," observes Dr. Gregory, "by a series of connected propositions, demonstrated that which is justly reckoned the most important in the theory of Statics, and which is now commonly spoken of under the title of the *Parallelogram of forces*. The demonstration here given is commenced upon the same principle (9) as that proposed by *D'Alembert*, in the Memoirs of the French Academy for 1769: it was somewhat simplified by *Francœur* in his Mechanics; but what is here offered, at the same time that it is more concise than the demonstration of Francœur, is freed, it is hoped, from some objectionable positions into which that author has certainly fallen."

The foregoing proposition of the parallelogram of forces, has also been established by some eminent mathematicians by processes purely analytical, and deduced from some obvious principle necessarily involved in the question itself, as for example, that if two equal forces P, P concur at an angle θ, the direction of the resultant must bisect this angle, and moreover its intensity must be some function

of P and θ, which is no more than saying that the intensity of the resultant must in some way depend on the intensity of its equivalent components and on their mode of action. This is the condition from which *Poisson* sets out his analytical investigation. We have given it with some little modification at the end of the volume.

(14.) From what has now been proved, it follows that when the intensities and directions of any two concurring forces are given, the determination of the third force equilibrating these will be reduced to the determination of the diagonal of a parallelogram, from having the two sides and included angle given; or still more simply, it will be reduced to the determination of the third side of a triangle, from having two sides and the included angle given; for if PC, PC′ are the two given forces (fig. 7,) then, by drawing CR equal and parallel to PC′, the diagonal PR will be just as well determined as if we had constructed the parallelogram CC′, so that instead of the sides and diagonal of a parallelogram, we may always represent three equilibrating forces as well in direction as in intensity by the three sides of a triangle taken in order as PC, CR, RP; so that as these sides are as the sines of their opposite angles, we may say that *the intensity of any one of three equilibrating forces is proportional to the sine of the angle between the directions of the other two*. If we represent the sides PC, PC′ by P, Q, and the angle CPC′ of their direction by a, then, since in the triangle CRP the angle C is the supplement of a, we shall have this analytical expression for R, viz., $R^2=P^2+Q^2+2PQ\cos.a$, so that the fundamental theorem of statics, when expressed algebraically, is precisely that which is also the fundamental theorem of plane trigonometry.

(15.) Knowing how to compound two forces, we may easily compound several or determine a single force which will balance them, and that either by geometrical construction or by analytical representation. Thus, suppose four forces PC_1, PC_2, PC_3, PC_4, concurred at P, then we might proceed geometrically as follows: Draw in the plane of the two forces PC_1, PC_2, the line C_1R_1, equal and parallel to PC_1, then PR_1 will, from what has already been proved, be the equivalent of PC_1, PC_2, and may therefore be substituted for them in the system. Again, in the plane of the two forces PR_1, PC_3, draw R_1R_2 equal and parallel to PC_3, then, as before, PR_2 will be the equivalent of PR_1, PC_3, that is of the three forces PC_1, PC_2, PC_3. Lastly, in the plane of the two forces PR_2, PC_4, draw R_2R_3, parallel and equal to PC_4, and we shall then determine PR_3, which must be the equivalent of the whole system. From this construction it is obvious, that if commencing at the point of concurrence P we draw successively PC_1, C_1R_1, R_1R_2, R_2R_3, equal and parallel to the several forces, then the line PR_3, which closes the polygon $PC_1R_1R_2R_3P$, will represent the resultant of

the system, and this in whatever planes the competent forces act, since in our construction we have not confined these forces to any particular planes. Should the concurring forces be in equilibrium, the last of the points R_1, R_2, R_3, &c. will fall on P, making the resultant 0.

This graphical method of compounding forces, will not, however, answer the purposes of computation, and we shall therefore now seek a general analytical expression for the resultant of any number of concurring forces, confining ourselves first to those only which are situated in one plane.

Determination of the Resultant of any Number of concurring Forces situated in one Plane.

(16.) We have already seen how any two concurring forces may be compounded into one, but before we can conveniently compound a greater number we must first reverse this process, and know how to *resolve* any single force into two concurring forces acting in certain proposed directions.

Let PR (fig. 8,) represent any given force acting on P, and let it be required to resolve it into two others concurring in P, and acting in the directions PC, PC′, so that PC may make a given angle α with PR, and PC′ may make a given angle β with PR. Parallel to PC′ draw RC, then (13) making PC′ equal to RC, PR will represent the force of which PC, PC′ represent the components, that is, PC, PC′ will be the components sought. Their analytical values will be obtained by determining trigonometrically the two sides PC, CR of the triangle CPR, from having the side PR and interjacent angles given; we see, therefore, that *the resolution of a given force in any two arbitrary directions may always be effected.* That it may be effected in the easiest way possible, it is requisite that the proposed directions be perpendicular to each other, for then the analytical operations are expressed simply by the equations, $PC = PR\cos.\alpha, PC' = PR\cos.\beta$, where α and β are complements of each other. When, therefore, we have to decompose a force into two others, and are at liberty to choose their directions, we shall always on the score of simplicity, assume these directions perpendicular to each other.

(17.) Let it now be required to compound into a single resultant the several concurring P, P_1, P_2, P_3, &c. (fig. 9).

Assume any rectangular axis AX, AY, the first AX making with the direction of the several forces the angles α, α_1, α_2, α_3, &c., and the second AY making with the same directions the angles β, β_1, β_2, β_3, &c. Then by resolving each of the proposed forces into two others acting along the assumed axes, we shall have for the sum X of all the components Ax, Ax_1, Ax_2, &c. acting along Ax,

$$P \cos. \alpha + P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + P_3 \cos. \alpha_3 + \&c. = X \ldots . (1),$$

and for the sum Y of all the other components, that is those acting along AY

$$P \cos. \beta + P_1 \cos. \beta_1 + P_2 \cos. \beta_2 + P_3 \cos. \beta_3 + \&c. = Y \ldots . (2),$$

so that these two sums, that is the single force X acting along AX and the single force Y acting along AY, may be substituted for the proposed system of forces; hence the resultant of these two will be the resultant of the original system. But the resultant R of two forces X and Y acting at right angles, being the diagonal of the rectangle XY, is $R = \sqrt{X^2 + Y^2} \ldots .$ (3), this, therefore, is the general expression for the resultant of any system of concurring forces acting in a plane.

(18.) It might at first sight appear, that since the angles β, β_1, &c., are the complements of α, α_1, &c., it would be better to write in the first member of (2) the expressions sin. α, sin. α_1, &c. instead of cos β, cos. β_1, &c.; such, however, is not generally the case, although this change would do very well for the particular arrangement of the forces exhibited in the figure, for it is easy to see that this arrangement is such as to cause all the components acting along each axis to conspire. If one of the forces P_3 were directed out of the angle YAX, as in fig. 10, its component Ax_3 would obviously oppose the conspiring forces Ax, Ax_1, Ax_2, and therefore, agreeably to (6), its analytical representation should carry a contrary sign, which it actually does do when we give it the form $P_3 \cos \alpha_3$, but not when written P_3 sin. β, although each form represents the same linear magnitude; similar remarks apply when P_3 is situated in either of the angles XAY′, X′AY′. Regarding then the components which act in the directions AX, AY as positive, and those which act in the opposite directions, AX′, AY′ as negative, the several terms in (1) and (2) will have their proper signs involved in those of their cosines.

(19.) In order to completely determine the resultant R, we must know its direction as well as its intensity (3). Now putting a, b, for its inclination to Ax, AY, we know that

$$X = R \cos. a,\ Y = R \cos. b \mathrel{.\,\dot{.}\,} \cos. a = \frac{X}{R},\ \cos. b = \frac{Y}{R},$$

either of which equations makes known the direction of the resultant, so that the resultant will be completely represented by the equations

$$\left. R = \sqrt{X^2 + Y^2}, \cos. a = \frac{X}{R}, \right\} \ldots . (4).$$

If the proposed forces are themselves in equilibrium, then R=0, so that we must then have $\sqrt{X^2 + Y^2} = 0$ or $X^2 + Y^2 = 0$; but as every

square is essentially positive, the sum of two cannot be 0 unless each separately is 0, so that when the forces are in equilibrium, we must have X=0, Y=0, showing that each system of components must be in equilibrium, and this is obviously true whatever be the inclination of the axes of the components.

(20.) As a particular example of the preceding general theory, let us suppose four forces, P, P_1, P_2, P_3, concurring in a point A, of which the intensities are respectively denoted by the numbers 2, 3, 4, 5, and let the angles included by their directions be $PAP_1=30°$, $P_1AP_2=15°$, $P_2AP_3=75°$. First assume, as above directed, two rectangular axes AX, AY, and, as their position is arbitrary, let us for greater simplicity suppose one of them AX to coincide with AP; then the inclinations of the several forces to the assumed axes will obviously be as follows:

$$\begin{array}{ll}
PAX=\ \ 0°=\alpha\ \therefore & \cos.\ \alpha\ =1 \\
P_1AX=\ 30°=\alpha_1 & \cos.\ \alpha_1=\tfrac{1}{2}\sqrt{3} \\
P_2AX=\ 45°=\alpha_2 & \cos.\ \alpha_2=\tfrac{1}{2}\sqrt{2} \\
P_3AX=120°=\alpha_3 & \cos.\ \alpha_3=-\tfrac{1}{2}\sqrt{3}
\end{array}$$

$$\begin{array}{ll}
PAY=90°=\beta\ \therefore & \cos.\ \beta\ =0 \\
P_1AY=60°=\beta_1 & \cos.\ \beta_1=\tfrac{1}{2} \\
P_2AY=45°=\beta_2 & \cos.\ \beta_2=\tfrac{1}{2}\sqrt{2} \\
P_3AY=30°=\beta_3 & \cos.\ \beta_3=\tfrac{1}{2}\sqrt{3}
\end{array}$$

consequently, the two general equations

$$P\cos.\alpha+P_1\cos.\alpha_1+P_2\cos.\alpha_2+\&c.=X$$
$$P\cos.\beta+P_1\cos.\beta_1+P_2\cos.\beta_2+\&c.=Y$$

become in this case

$$2+\tfrac{3}{2}\sqrt{3}+2\sqrt{2}-\tfrac{5}{2}\sqrt{3}=X$$
$$0+\tfrac{3}{2}+2\sqrt{2}+\tfrac{5}{2}\sqrt{3}=Y;$$

hence, the numerical values of X and Y being thus determined, the value of $R=\sqrt{x_2+y_2}$ is known, and thence also of

$$\cos.\ a=\frac{X}{R},\ \text{or}\ \cos.\ b=\frac{Y}{R}.$$

Determination of the Resultant of any Number of concurring Forces situated in different Planes.

(21.) It was necessary before we could compound together several forces acting in one plane, first to determine the resultant of two, or to establish the parallelogram of forces, so likewise before we can treat the more general case, or compound together forces acting in different planes, we must first know how to determine the resultant of three. This however is a very easy matter, it has indeed

been accomplished geometrically already (15), and not only for three but for any number of forces. But let there be three P, P_1, P_2, concurring in A, (fig. 11,) and represented by the lines AB, AC, AD; then we know from the article just referred to, that if we draw BE in the same plane with, and equal and parallel to, AC, and then EF in the same plane with, and equal and parallel to, AD, the line AF, which closes the twisted quadrilateral ABEFA, will represent the resultant. Now the three lines AB, BE, EF, are obviously the three edges of a parallelopiped BG, of which AF is the diagonal; hence the lines representing three *concurring forces not in the same plane form the edges of a parallelopiped whose diagonal is their resultant.* It is manifest that if any force AF be proposed, and we draw from A three lines AB, AC, AD, in any directions whatever, not all in the same plane, nor yet any two in the same plane as AF, we may construct a parallelopiped having these three lines for edges, and AF for its diagonal, the opposite edges meeting in F. Now the forces represented by the edges of this parallelopiped meeting in A, have the given force AF for their resultant, therefore this given force has these three for its components, so that *any force may be decomposed into three concurring forces acting in any proposed directions, provided all three are not in one plane nor any two in the same plane as the proposed force.*

This decomposition will be most easily effected, analytically, when the directions of the components are at right angles; for if α, β, and γ, represent the inclinations of the proposed force R to these several rectangular directions, then the three components will be obviously expressed by R cos. α, R cos. β, R cos. γ . (1), for in fact, these components are no other than the projections of the original force on three rectangular axes, calling these several projections or component forces, X, Y, Z, we have, by adding their squares, $X^2+Y^2+Z^2=R^2(\cos.^2\alpha+\cos.^2\beta+\cos.^2\gamma)$,
but (*Anal. Geom.* p. 228, $\cos.^2\alpha+\cos.^2\beta+\cos.^2\gamma=1$ . . (2),

$$\therefore R=\sqrt{X^2+Y^2+Z^2}$$

It thus appears that when we wish to decompose a given force R in three rectangular directions AX, AY, AZ, making with R any proposed angles α, β, γ, we shall have, for the analytical values of the components, the expressions

$$X=R\cos.\alpha,\ Y=R\cos.\beta,\ Z=R\cos.\gamma\ .\ .\ .\ .\ (3),$$

and when on the other hand we wish to compound three given rectangular forces X, Y, Z, the *intensity* of the resultant will be given by the expression $R=\sqrt{X^2+Y^2+Z^2}$ . . . . (4),
and its *direction* by the expressions

$$\cos.\alpha=\frac{X}{R},\ \cos.\beta=\frac{Y}{R},\ \cos.\gamma=\frac{Z}{R}\quad (5).$$

Two of these latter equations are however sufficient to fix the position of the resultant; since, on account of the necessary condition (2), any one of the cosines become fixed when the other two are; thus, when α and β are determined, we get γ by the equation $\cos.\gamma=\sqrt{1-\cos.^2\alpha-\cos.^2\beta}$ . . . . (6).
We need not embarrass ourselves here with any inquiry about the ambiguity of the sines of the cosines in (5), arising from the ambiguity of the sine of the radical R in (4); for, as we know that the resultant must necessarily lie within the angle formed by X, Y, Z, the angles α, β, γ, which it forms with these lines, must always be acute.

(22.) Let us now proceed to determine the resultant of any number of concurring forces P, P_1, P_2, P_3, &c., situated in space, and acting in given directions.

Through the point of concourse A draw three rectangular axes AX, AY, AZ, and let us call

α, β, γ, the angles which P makes with these axes,
α_1, β_1, γ_1, ... P_1
α_2, β_2, γ_2, ... P_2
&c. &c.

then, by decomposing each force according to these axes, we have

$P\cos.\alpha$, $P\cos.\beta$, $P\cos.\gamma$ for the components of P
$P_1\cos.\alpha_1$, $P_1\cos.\beta_1$, $P_1\cos.\gamma_1$ . . . . P_1
$P_2\cos.\alpha_2$, $P_2\cos.\beta_2$, $P_2\cos.\gamma_2$ . . . . P_2
&c. &c.

Adding together all the forces which act in each axis, the three sums X, Y, Z, will represent three rectangular forces acting in given directions, which may be substituted for the proposed system, the values of these three forces being

$$\left.\begin{array}{l} P\cos.\alpha+P_1\cos.\alpha_1+P_2\cos.\alpha_2+\&c.=X \\ P\cos.\beta+P_1\cos.\beta_1+P_2\cos.\beta_2+\&c.=Y \\ P\cos.\gamma+P_1\cos.\gamma_1+P_2\cos.\gamma_2+\&c.=Z \end{array}\right\} \ldots . (1).$$

Having thus reduced the system of forces to three, we have, for the intensity of the resultant, the expression $R=\sqrt{X^2+Y^2+Z^2}$. (2), and for the angles a, b, c, which it makes with the axes, the expressions $\cos.a=\frac{X}{R}$, $\cos.b=\frac{Y}{R}$, $\cos.c=\frac{Z}{R}$ . . . . (3).

In this way, therefore, we may completely determine the resultant of any system of forces situated in space, when we know the intensity of each force, and the angles its direction makes with three rectangular axes. The cosines in (1) necessarily carry with them the proper sines as in art. (18).

When the system of concurring forces is in equilibrium, then, since $R=0$, we must have $X^2+Y^2+Z^2=0$; but as every square is

essentially positive, this cannot be unless, $X=0$, $Y=0$, $Z=0$, that is to say, we must have (equa. 1,)

$$\left.\begin{array}{l} P\cos.\alpha+P_1\cos.\alpha_1+P_2\cos.\alpha_2+\&c.=0 \\ P\cos.\beta+P_1\cos.\beta_1+P_2\cos.\beta_2+\&c.=0 \\ P\cos.\gamma+P_1\cos.\gamma_1+P_2\cos.\gamma_2+\&c.=0 \end{array}\right\}\ldots\ldots(4).$$

These, therefore, are the *equations of equilibrium* of a system of concurring forces P, P_1, P_2, &c situated any how in space.

(23.) The projection of any line in space on two rectangular axes, may obviously be found by first projecting the line on the plane of those axes, and then projecting this projection on the axes themselves. Hence, if a system of forces in equilibrium be projected on the plane of xy, the forces represented by these projections will be also in equilibrium, seeing that the components of these forces will be $X=0$ and $Y=0$. Now the position of one of the rectangular planes, as the plane of xy, is always arbitrary; on this account therefore, and on account of the equations (4) it follows, that *when any number of forces are in equilibrio, their projections upon any plane or upon any line will also be in equilibrio.*

(24.) As the resultant R or AF (fig. 11,) of any system of forces is the diagonal of the parallelopiped whose edges are the components X, Y, Z, or AB, AC, AD, it follows that X, Y, Z, are no other than the co-ordinates of the point F, (*Anal. Geom.* p. 222.) Knowing, therefore, the co-ordinates of a point F in the line AF passing through the origin A, we know enough to enable us to write the equation of this line or of the resultant. These equations are (*Anal. Geom.* p. 226-7,) $z=\frac{Z}{X}x$, $z=\frac{Z}{Y}y$. If we remove the origin of the rectangular axes without altering the directions of the axes, so that the co-ordinates of the point of concourse A may be x', y', z', then the equation of the resultant will be $z-z'=\frac{Z}{X}(x-x')$, $z-z'=\frac{Z}{Y}(y-y')$, or the line will be equally represented by combining either of these equations with that of the third projection, viz. $x-x'=\frac{X}{Y}(y-y')$.

Having now established the general theory of the equilibrium of a free point acted upon by any number of forces any how situated, we shall devote a short chapter to the application of this theory to particular examples.

CHAPTER II.

PROBLEMS ILLUSTRATIVE OF THE PRECEDING THEORY.

(24.) We have already adverted (4) to the propriety of representing balanced forces by means of weights, from the circumstance that any influence tending to move a free point may be always counteracted and rendered nugatory by the opposing influence of some weight communicating to the point, by means of a cord attached to both. All, therefore, that has been established in the preceding chapter respecting the equilibration of forces acting on a point, applies when these forces are weights acting on a point through the intervention of cords, provided only that we consider these cords to be themselves without weight or thickness, to be inextensible, and to be perfectly capable of moving over the fixed points or pullies, (as at fig. 12,) employed to direct the influence of the weights, with perfect freedom. The consideration of a system of weights thus acting on a free point through the intervention of cords and fixed points, introduces the consideration of two new modifications of force, viz. *Tension* and *Pressure*.

By the tension of a cord is to be understood its tendency to stretch under the influence of an appended weight; as this tendency varies with the weight, this is taken as its measure; so that when we speak of the tension of a cord as a force, we always mean the weight which produces that tension. The pressure on a fixed point is measured by, or is equal to, that force which must be applied to it, when supposed free, to keep it in equilibrium.

Problem I.—A cord PACBP, passes over two fixed points or small pulleys A, B in the horizontal line AB, and two given equal weights, suspended at the extremities P, P, support a third given weight W. It is required to determine the position of C, (fig. 12).

The point C is kept in equilibrium by the equal tensions (P or P_1), of the strings CA, CB, and by the weight W acting vertically. Hence, by resolving the forces in the directions of two axes CX, CY, the one parallel and the other perpendicular to AB, the forces in each axis must destroy each other. Hence, taking first the components in CX, we have the condition

$$P \cos. \alpha + P_1 \cos. \alpha_1 = 0 \text{ or } P \cos. \alpha - P_1 \cos. \alpha' = 0,$$

from which we immediately infer that as $P = P_1$, $\alpha = \alpha'$, and therefore $\beta = \beta_1$, so that the triangles ACY, BCY are equal, and CY bisects AB. Again, because $\beta = \beta_1$, the components in CY are $2P \cos. \beta$, and W in the opposite direction, therefore the second condition is

$$2P \cos.\beta - W = 0 \therefore \cos.\beta = \sin. B = \frac{W}{2P}.$$

This equation is sufficient to determine the point C or the line EC; but, to avoid any trigonometrical computation, let us put for cos. β its equal $\frac{YC}{BC}$, then $\frac{BC}{YC} = \frac{\sqrt{BY^2+YC^2}}{YC} = \frac{2P}{W}$

$$\therefore \frac{BY^2}{YC^2} = \frac{4P^2-W^2}{W^2} \therefore YC = \frac{W}{\sqrt{4P^2-W^2}} BY.$$

The solution of this problem may be conducted differently as follows: Parallel to CB draw AE, and produce the perpendicular CY till it meets in E; then the triangle ACE thus constituted will have its three sides in the directions of the three balancing forces, and will, therefore, be proportional to them; hence, as the tensions of CA, CB are equal, the sides CA, AE are equal, so that the perpendicular AY bisects CE; also

$$P : W :: AC : EC = 2YC \therefore \frac{2P}{W} = \frac{AC}{YC}$$

and the remainder of the solution may be as above.

If we suppose $W=0$, then the above expression for YC shows that $YC=0$, as it obviously ought to be; that is, the cord will be brought into a horizontal line AB. But upon no other hypothesis will this be the case, except, indeed, we suppose the weights P to be infinitely great, for however small W be assumed, yet so long as P is of finite magnitude YC will have a finite value, and can never be accurately 0, so that it is impossible for any two weights P, P_1, however great, acting as in the figure, to draw a third weight W ever so small up to the horizontal line AB. The same is true if instead of a small weight W attached to a cord without weight, we consider the cord itself to have weight: we may therefore say with Professor Whewell

> "Hence no force, however great,
> Can stretch a cord, however fine,
> Into a horizontal line
> Which shall be accurately straight."

As W increases from 0, YC increases and becomes infinite when $W=2P$, so that when the weight W is either equal to or greater than 2P, there can be no equilibrium, for W will continually descend drawing up the weights P, P_1.

PROBLEM II.—Suppose the weights P, P_1, are unequal, and that the line AB, instead of being horizontal, makes a given angle BAB′ with the horizontal line AB′: to determine the position of C, (fig. 13.)

The solution will perhaps be most easily obtained without resolving the forces; thus, draw AE parallel to CB, meeting the vertical line CE in E, then the three sides CA, AE, EC, being in the directions of the three equilibrating forces P_1, P, W, are proportional to them; hence, knowing the proportion of the three sides of the triangle AEC, we may determine its angles: we may, therefore, consider the angles ACE and AEC=ECB as found; consequently the angle CAD, the complement of ACE, becomes known, and DAB being given the angle CAB becomes known; hence, in the triangle CAB, we have the side AB and the angles C and A to determine the two sides AC, BC.

Thus, suppose P=4℔, P_1,=3℔, and W=5℔: also AB=6 feet, and the angle DAB=30°, then the angles of a triangle whose sides are AC=3, AE=4, EC=5, are CAE=90°=BCA, AEC=36° . 54′=ECB; hence DCA=90°—ECB=53° . 6′, and consequently ABC=DCA—30°=23° . 6′. We thus have AB=6 feet, ACB=90°, ABC=23° . 6′, whence AC=2 · 354, and BC=5 · 518 feet.

PROBLEM III.—Two equal weights P, P_1, balance themselves over any number of fixed pulleys: to determine the pressure on each, (fig. 14.)

Each of the points A, B, C, &c. are kept in equilibrium by three forces, viz. by the equal tensions on each side of it and by the pressure it sustains, which latter is therefore equal and opposite to the resultant of the two equal tensions P. Hence, calling the angles at A, B, C, &c. a, b, c, &c. we have

$$\begin{aligned} \text{the pressure on } A &= 2P\cos.\tfrac{1}{2}a \\ B &= 2P\cos.\tfrac{1}{2}b \\ C &= 2P\cos.\tfrac{1}{2}c. \\ \&c. \quad & \quad \&c. \end{aligned}$$

PROBLEM IV.—A cord ACB of given length is fastened to two hooks A and B, and a weight W is at liberty to slide by means of the ring C freely upon this cord; at what point will it rest? (fig. 15.)

It is obvious that if the two hooks were in a horizontal line, as A′, B the weight W ought to settle itself at a point symmetrically situated with respect to the two points A′, B, that is CA′B will be an isosceles triangle, and therefore the angles A′CH, BCF, made with the horizontal line HF, will be equal. But if the hook be at A instead of at A′, then the force in CA being the tension of CA, or the pressure upon the hook A, we may consider the hook to be removed and a force equivalent to this pressure to be applied at A; but it matters not at what point of its direction a force is applied, so that C will continue undisturbed if the force be applied at A′, that is, it will make no difference as to the position of C, whether the hook be

at A or at A′, the tension of CA being the same throughout; hence the angles ACH, BCF are equal, and the tension of CA equal to that of CB. Produce AC to meet the vertical ED in D, then the sides of the triangle BCD, being in the direction of the forces, are proportional to them; this triangle is moreover isosceles having CB =CD on account of the equal tensions of CB, CD, or of the equal angles BCF, ACH; hence AD is equal to the length of the string. We thus have given

$$AB=a,\ AD=l,\ EAB=\alpha\ \therefore\ AE=a\cos.\alpha,\ EB=a\sin.\alpha$$

$$ED=\sqrt{l^2-a^2\cos.^2\alpha}\ \therefore BF=\frac{\sqrt{l^2-a^2\cos.^2\alpha}-a\sin.\alpha}{2};$$

also, since, DE : EA : : BF : FC

$$\therefore FC=\frac{a\cos.\alpha}{2}\left\{1-\frac{a\sin.\alpha}{\sqrt{l^2-a^2\cos.^2\alpha}}\right\},$$

these values of BF, FC determine the point C,

As to the pressure p on the hook B or A, we have

$$BF : BC : : \tfrac{1}{2}W : p, \text{ but } BF : BC : : ED : DA$$

$$\therefore p=\frac{DA}{2\,ED}W=\frac{l}{2\sqrt{l^2-a^2\cos.^2\alpha}}W.$$

Or the pressure may be found by first determining the angle D by means of the sides AE, AD; this angle being equal to BCY or ACY, we have, by calling it β, and resolving the equal tensions p along the axis CY, $2p\cos.\beta=W\ \therefore\ p=\dfrac{W}{2\cos.\beta}$.

In the very same way the problem may be solved, when, instead of a weight W acting vertically, any power acting obliquely be attached to the ring, for the lines BA′, AE being drawn perpendicular to the direction of this force, and ED parallel to it, the above reasoning becomes then obviously applicable to this case.

The question may be viewed in rather a different manner from that above, by considering that as the cord ACB is of constant length, the point C, before arriving at a state of rest, must describe the arc of an ellipse, and, moreover, that the place of rest must be at the lowest point possible; hence the horizontal line HF must be a tangent at C to the ellipse whose foci are A and B, seeing that every other point in this ellipse is above that line; hence, by the property of the ellipse, the angles BCF, ACH are equal, and, consequently, the tensions are equal, because their components in the directions CF, CH must be equal.

It is very clear, although we have not assumed it above, that because the cord passes *freely* through the ring, the tension of the part CB must be communicated to the part CA, for nothing hinders this communication, so that the cord will be equally tense throughout.

PROBLEM V.—A cord of given length passes over two pullies, and one of its extremities P is put through a small ring or noose at the other extremity C; a given weight W is then attached to P: it is required to determine the tension of the string when in equilibrium, as also how much of the cord will hang below the ring, (fig. 16.)

The tension of CP is measured by the weight W, and this same tension must be communicated to the parts CB, BA, AC, since the cord passes freely through the ring and over the pullies; but when three equal concurring forces are in equilibrium, the angles formed by their directions are each 120°; hence, drawing the horizontal line AD, we have in the isosceles triangle ADC, the angles A, D, each equal to 30°, and the angle C equal to 120°; consequently, as the position of AB, with respect to the horizon, that is, the angle BAD, is known, we know in the triangle BAC the side AB and the angles A and C which is sufficient for the determination of AC, BC, and, therefore, of the place of C; also the perimeter of this triangle being taken from the whole length of the string, leaves CP the distance of the weight from the noose.

If, instead of the loop or ring at C, the extremity C were firmly fastened by a knot to BCP at a given distance from the other extremity P, then the tension of CP would not be freely communicated to CA or CB, but whatever tension CA had, the same would be communicated to AB and BC, for the rope being freely moveable over A and B, could not rest so long as either of these tensions prevailed. The equal tensions of CA, CB, as also the position of the knot C, will be determined in this case as in problem IV.

PROBLEM VI.—The extremities of a given cord are fastened to two hooks given in position, and to a given point in it is applied a power P acting in a given direction: to determine the pressure upon the hooks (fig. 17.)

As the point C in the given cord ACB is given, as also the line AB, therefore the three sides of the triangle ABC are given, and, consequently, the three angles; also, as the direction of YCP is given, the angles at Y are given; hence the two exterior angles ACP, BCP are given. If, therefore, we draw CX perpendicular to CY, the angles a, a' will be known, so that calling the pressures on B and A, p and p', we shall have, by the conditions of equilibrium,

$$p \cos. a - p' \cos. a' = 0, \quad p \sin. a + p' \sin. a' = \text{P},$$

$$\therefore p = \frac{\text{P} \cos. a'}{\cos. a \sin. a' + \sin. a \cos. a'} = \frac{\text{P} \cos. a'}{\sin. \text{ACB}}$$

$$p' = \frac{\text{P} \cos. a}{\cos. a \sin. a' + \sin. a \cos. a'} = \frac{\text{P} \cos a}{\sin. \text{ACB}}.$$

The pressures p, p', therefore, are to each other as the sines of the

angles β_1, β, or of the angles ACP, BCP; but this much we might immediately have inferred from the property that when three forces equilibrate, each is proportioned to the sine of the angle between the directions of the other two.

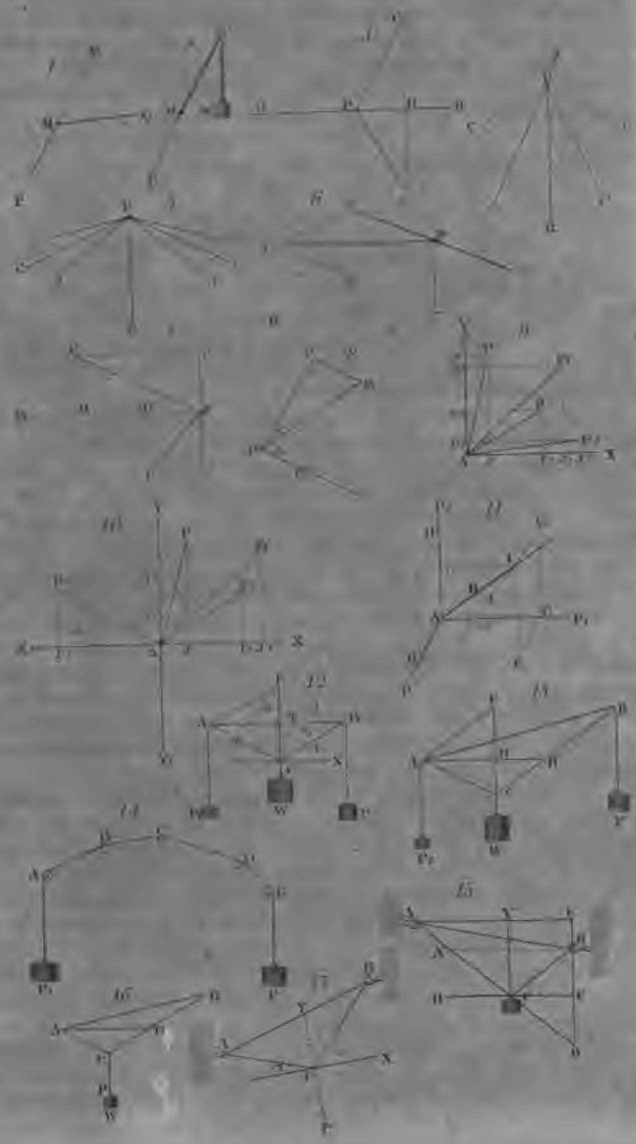
CHAPTER III.

ON THE FUNICULAR POLYGON AND CATENARY.

(25.) If a cord be kept in equilibrium by means of several forces P, P_1, P_2, P_3, &c. acting at the knots p_1, p_2, p_3, &c. the figure p_1, p_2, p_3, &c, which it forms itself into, is called the *funicular polygon* (fig. 18). We propose here to investigate the conditions of equilibrium of such a figure.

And first it is obvious, that the several points p_1, p_2, &c. are each kept in equilibrium by the three forces which concur there; it is equally obvious that the tension of the string p_1, p_2 being the same throughout, there is the same pressure upon the knot p_2, as upon the knot p_1, but exerted in the opposite direction, and the same of any two consecutive knots p_2, p_3; p_3, p_4, &c. Hence, if the three forces which equilibrate p_1, were applied to p_2, the equilibrium of p_2 would remain undisturbed; but of the forces thus acting on p_2 two would destroy each other, since, as just observed, the tension of $p_1\,p_2$ presses the points p_1, p_2 with equal force but in opposite directions; we may, therefore, consider but four forces acting on p_2, viz. the forces in p_2 P_2, and in $p_2\,p_3$ together with those in p_1 P, and in p_1 P_1. Again, if the four forces equilibrating p_2, be transferred to p_3, the equilibrium of p_3 will remain undisturbed, and, as before, the two forces due to the tension of $p_2\,p_3$ will destroy each other, and thus the point p_3 will be kept in equilibrium by five forces of which only one, viz. that in $p_3\,p_4$ will be a tension, the others being the forces P, P_1, P_2, P_3, originally applied to the cord at the knots p_1, p_2, p_3. Proceeding in this way from knot to knot, it is plain that when we shall have arrived at the last knot or at the extremity of the cord that the point will be kept in equilibrium by the concurrence of all the forces originally distributed along the cord, at the points p_1, p_2, &c. the directions of these forces being preserved.

From all this it follows then, that for the funicular polygon to exist, the intensity and directions of the several forces acting at the knots, must be such, that if they were all applied to one point they would keep it in equilibrium, and that to find the direction and tension of the n th side of the polygon it will only be necessary to as-



certain what would be the direction and intensity of the resultant of all the forces acting on the $n-1$ preceding knots, if they were all to concur.

It thus appears, that in the funicular polygon the conditions of equilibrium are the very same as if the forces all concurred, or were to be transferred parallel to themselves to a single point, so that if we assume three rectangular axes, and call, as before, the angles at which the several directions of the forces are inclined to these, α, α_1 α_2, &c. β, β_1, β_2, &c. γ, γ_2, γ_3, &c., the conditions necessary to the existence of the funicular polygon will be

$$\left.\begin{array}{l} P\cos.\alpha+P_1\cos.\alpha_1+P_2\cos.\alpha_2+\&c.=0 \\ P\cos.\beta+P_1\cos.\beta_1+P_2\cos.\beta_2+\&c.=0 \\ P\cos.\gamma+P_1\cos.\gamma_1+P_2\cos.\gamma_2+\&c.=0 \end{array}\right\} \quad . \ (1).$$

When the forces all act in one plane, then two axes taken in this plane will be sufficient, as one of these equations then becomes identically 0, the conditions being

$$\left.\begin{array}{l} P\cos.\alpha+P_1\cos.\alpha_1+P_2\cos.\alpha_2+\&c.=0 \\ P'\cos.\beta+P_1\cos.\beta_1+P_2\cos.\beta_2+\&c.=0 \end{array}\right\} \ . \ (2).$$

To construct the polygon in any particular case, we must know not only the intensities and directions of the several forces, but also their points of application; when these are known, we may easily construct the successive sides of the polygon; thus the resultant of P, P_1 being determined, we shall thence have the direction and intensity of the force in $p_1\,p_2$, and the point p_2 being known we thus have the side $p_1\,p_2$; in like manner the resultant of P_2 and the force in $p_2\,p_1$, just determined, will make known the intensity and direction of the force in $p_2\,p_3$, so that, as the point p_3 is given, we may construct the second side $p_2\,p_3$, and so on till the polygon is completed.

If any one of the forces were attached to the cord not by a fixed knot, as we have hitherto supposed, but by a moveable ring, then in the equilibrated state of the system the tensions on each side of the ring would be equal, and would therefore form equal angles with the force on the ring.

In this way may the absolute tension between any two proposed knots be determined, but if we wish merely to find the ratio of the tensions of any two sides of the polygon, then, recollecting that each of three equilibrating forces acting on a point is proportional to the sine of the angle between the other two, and calling the several tensions t, t_1, t_2, &c. (see fig. 18,) we have

$$\frac{t}{t_1}=\frac{\sin.a_1}{\sin.a},\ \frac{t_1}{t_2}=\frac{\sin.a_3}{\sin.a_2},\ \frac{t_2}{t_3}=\frac{\sin.a_5}{\sin.a_4},\ \&c.$$

Multiplying these equations together, and omitting the factors common to both numerator and denominator, we have generally

$$\frac{t}{t^n}=\frac{\sin. a_1 \sin. a_3 \sin. a_5 \ldots \sin. a_{2n-1}}{\sin. a \sin. a_2 \sin. a_4 \ldots \sin. a_{2n-2}} \ldots (3).$$

Should, therefore, the angles a, a_1, a_2, &c. be equal throughout, the tension will be uniform throughout, and, conversely, if the tension be uniform throughout, the angles must be all equal; when, therefore, the angles are equal, the uniform tension of the cord is measured by either of the extreme forces P, P_n, which are necessarily equal in intensity, as they measure the equal tensions of the extreme sides of the polygon.

(26.) The most important case of the funicular polygon is that in which the several forces P_1, P_2, &c. (fig. 19,) are weights, acting in the same vertical plane upon fixed points of the cord when suspended at its two extremities P, P_n, (fig. 19), we shall therefore consider this case in particular, and first we may remark, that the polygon so formed will lie wholly in the vertical plane of the forces, for of the three equilibrating forces concurring in p_1, two, viz. those in p_1, P, and in p_1 P_1, are in the vertical plane; therefore the third, or that in p_1, p_2, must be in the same plane; also, this last and that in p_2 P_2 being in the vertical plane, the force in p_2 p_3 must be in that plane, and so on. Let us then draw in this plane the horizontal and vertical axes PX, PY; the angles α_1, α_2, &c. which the directions of the forces make with the first of these axes, are each equal to 90°, and the angles β_1, β_2, &c. made with the other axis, are each 0; hence, denoting the sum of all the weights P_1, P_2, &c. by W, the equations of equilibrium (2) at page 35 become, in this case,

$$\left.\begin{array}{l} P\cos.\alpha+P_n\cos.\alpha_n=0 \\ P\cos.\beta+P_n\cos.\beta_n+W=0 \end{array}\right\} \ldots (1),$$

where P and P_n are the pressures on the two points of suspension; these pressures are therefore readily determinable if the angles α, α_n, that is the directions of the extreme cords are given, there being no necessity to know the situation of the knots, nor yet the separate forces P_1, P_2, &c., but only their sum W. All this, indeed, may be determined from the equations themselves; thus, let t represent the tension of any side of the polygon, and let us put a for the angle it makes with the horizontal axes, and b the angle it makes with the vertical axis.

The tension t may be considered as exerting a pressure upon the knot at that extremity of the proposed side which is farthest from the point P, so that substituting this pressure for P_n in the equations (1) and calling the sum of the weights between P and this knot w, we have

$$\left.\begin{array}{l} P\cos.\alpha+t\cos.a=0 \\ P\cos.\beta+t\cos.b+w=0 \end{array}\right\} \ldots (2),$$

two equations from which the two unknowns t and a may be deter-

mined, and thus the intensity and direction of the force in any side of the polygon ascertained; and, from knowing the intensities and directions of the forces in two adjacent sides, we find the intensity of the vertical force at the angle by taking the resultant.

As to the ratio of any two tensions, it is involved in the general expression (3) of last article, which, because in the case under consideration a_1 is the supplement of a_2, a_3 the supplement of a_4, and so on, reduces to $\frac{t}{t_n} = \frac{\sin. a_{2n-1}}{\sin. a}$ in like manner $\frac{t}{t_{n'}} = \frac{\sin. a_{2n'-1}}{\sin. a}$, $\therefore \frac{t_{n'}}{t_n} = \frac{\sin. a_{2n-1}}{\sin. a_{2n'-1}}$; so that *the tensions of any two sides of the polygon are reciprocally as the sines of the angles which they form with the vertical axis.*

Since these angles are the complements of those which the same sides form with the horizontal axis, we may substitute the cosines of these latter for the sines of the former, or because $\cos. = \frac{1}{\sec.}$ we may say that *the tensions are directly as the secants of their inclination to the horizon.*

(27.) *The Catenary.* By referring to equations (1), last article, we see that they express the conditions of equilibrium of these three forces acting at their point of concurrence, viz. the force P inclined at an angle α to the horizon, the force P_n inclined at an angle α_n, and the vertical force W. Hence, in our polygon (fig. 19), if we produce the directions of the pressures P, P_n, that is the extreme sides of the polygon, the point O in which they meet must be the point of concourse of which we speak, at which the vertical weight W and the pressures P, P_n acting maintains the equilibrium of O; these pressures are therefore the same as if all the weights acting at the angles of the polygon were collected and applied at O, it would therefore not be improper to consider W so applied as the resultant of all the original weights.

If we suppose in our polygon the weights to be attached at equal distances, the less these distances are taken the greater will be the number of sides of the polygon, and, consequently, the figure will approach the more nearly to a curvilinear form, which form it must actually assume when the distances between two consecutive weights become 0, that is, when the weights act upon every point of the cord; now this is the same as considering every point itself to be weighty, so that the curve, of which we speak, will be that which a perfectly flexible physical line or chain actually assumes, when suspended at its extremities. It is called the *catenary curve*, (fig. 20.)

The direction of the pressures on the points of suspension, will, obviously, be tangents to the catenary at those points, and, from

what has been said above, it appears that the point O, in which these directions meet, would be kept in equilibrium by the pressures or the tensions of the lines OP, OP_n, and by the whole weight W of the chain suspended at O.

(28.) Let us seek the equation of the catenary, supposing that the cord or chain is uniformly heavy throughout, that is that the lengths of any two portions are to each other as their weights. Taking the horizontal and vertical lines PX, PY, for axes of co-ordinates, we shall have for any point M, $Pm=x$, $mM=y$ and $PM=s$; and the portion s of the cord is held in equilibrium by the tensions at P and M, acting in the directions OP, OM of the tangents at those points and also by the weight of s; or the point O is held in equilibrium by the same tensions and the vertical weight s, the relation therefore between s and the tensions is determined by the relation between the sines of the angles about O, that is (14) $\frac{\sin. POM}{\sin. MOs}=\frac{s}{p}$, where p represents the pressure on P. Now

$$\begin{aligned}\sin. POM &= \sin. (IOM+POI)\\ &= \sin. IOM \cos. POI + \cos. IOM \sin. POI\\ &= \sin. mMO \sin. \alpha - \cos. mMO \cos. \alpha.\end{aligned}$$

consequently, since $mMO=MOs$

$$\frac{\sin. POM.}{\sin. MOs}=\sin. \alpha - \cot. mMO \cos. \alpha;$$

but (*Diff. Calc.* p. 113,) $\cot. mMO=\frac{dy}{dx}$ hence

$$\frac{s}{p}=\sin. \alpha - \frac{dy}{dx}\cos. \alpha \mathrel{.\,\cdot.} s=p\sin. \alpha - p\frac{dy}{dx}\cos. \alpha \ldots (1),$$

which is the differential equation of the catenary.

We may obtain a differential equation, involving only x and y, provided we differentiate this with respect to x and substitute for $\frac{ds}{dx}$ its equal $\frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}}$, for we thus have

$$\sqrt{1+\frac{dy^2}{dx^2}}=-p\cos. \alpha\frac{d^2y}{dx^2} \mathrel{.\,\cdot.} 1=-p\cos. \alpha\frac{\frac{d^2y}{dx^2}}{\sqrt{1+\frac{d^2y}{dx^2}}}.$$

The numerator of the fraction in the second member of this equation will obviously be the differential of the denominator if we multiply it by dy;* by doing this, therefore, and then integrating the

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* This is the same as multiplying both sides by $\frac{dy}{dx}$ and then multiplying by dx, to prepare each side for integration.

equation, we have $y = - p \cos.\alpha \sqrt{1 + \frac{dy^2}{dx^2}} + c$, from which we get $\frac{dy}{dx} = \frac{\sqrt{(c-y)^2 - p^2 \cos.^2 \alpha}}{p \cos.\alpha} \ldots (2)$.

It remains to determine the constant c, for which we have this condition, viz. that when $x=0$ and $y=0$, that is at the point P, $\frac{dy}{dx} = \tan.\alpha$, so that for this point the equation (2), just deduced, is

$$\tan.\alpha = \frac{\sin.\alpha}{\cos.\alpha} = \frac{\sqrt{(c^2 - p^2 \cos.^2 \alpha)}}{p \cos.\alpha},$$

or $p \sin.\alpha = \sqrt{c^2 - p^2 \cos.^2 \alpha}$

$$\therefore c^2 = p^2 (\sin.^2 \alpha + \cos.^2 \alpha) = p^2 \therefore c = p :$$

hence the differential equation of the catenary (2) is

$$\frac{dy}{dx} = \frac{\sqrt{(p-y)^2 - p^2 \cos.^2 \alpha}}{p \cos.\alpha} \ldots (3.)$$

If we substitute this value of $\frac{dy}{dx}$, in the equation (1), there results $s = p \sin.\alpha - \sqrt{(p-y)^2 - p^2 \cos.^2 \alpha}$. (4), which expression, for the length of the arc, proves that it is rectifiable.

In order to obtain an equation between x and y, independently of differentials, let us put in (3) $p - y = z$, $p \cos.\alpha = a$, then $dy = -dz$ and the equation reduces to

$dx = -a \frac{dz}{\sqrt{z^2 - a^2}} \ldots (5)$; to render this rational we must assume $\sqrt{z^2 - a^2} = z - z'$, from which we get the equation $2zz' = a^2 + z'^2$, which differentiated gives

$$zdz' + z'dz = z'dz' \therefore \frac{dz}{z - z'} = -\frac{dz'}{z'} = -d \log. z';$$

that is, from (5) $dx = ad \log. z' \therefore x = a \log. z' + c$ that is restoring the value of z', $x = a \log. \{z - \sqrt{z^2 - a^2}\} + c$; or, restoring the values of z and a

$$x = p \cos.\alpha \log. \{(p-y) - \sqrt{(p-y)^2 - p^2 \cos.^2 \alpha}\} + c \ (6).$$

The constant c may be determined from the condition that $x=0$ when $y=0$, the origin being at P, this condition gives $c = -p \cos.\alpha \log. \{p(1 - \sin.\alpha)\}$, and thus the equation (6) is

$$x = p \cos.\alpha \log. \left\{\frac{(p-y) - \sqrt{y^2 - 2py + p^2 \sin.^2 \alpha}}{p(1 - \sin.\alpha)}\right\} \ldots (7).$$

In order to determine the lowest point in the catenary, or that point at which the tangent of the inclination to the horizon is 0, we must put

$\frac{dy}{dx}=0$ in (3), which will give for y the value $y=p\ (1-\cos.\alpha)\ldots(8)$, and this put for y in the equation (7) gives

$x=p\cos.\alpha\log.\frac{\cos.\alpha}{1-\sin.\alpha}\ldots(9)$; also the length s of the cord hanging between the point of suspension P and the lowest point is, by equation, (4) $s=p\sin.\alpha\ldots(10)$.

(29.) If both points of suspension P, P_n, are in the same horizontal line, the portion of the cord between P and the lowest point will, obviously, be equal in length, and symmetrical in figure to that between P_n and the lowest point: hence the value of s in (10) will be half the length of the cord. When, therefore, we know the horizontal distance D of the points P, P_n and the length L of the cord, we may by help of the last three equations determine the angle α and the tension p: thus dividing (9) by (10) we have

$$\frac{D}{L}=\frac{\cos.\alpha}{\sin.\alpha}\log.\frac{\cos.\alpha}{1-\sin.\alpha};\ldots(11)$$

an equation from which, the unknown quantity α, may be determined by approximation, after which p will be given by (10), viz.

$$p=\frac{L}{2\sin.\alpha}\ldots(12);$$

knowing, therefore, α and p, we may readily find the tension t at any point of the cord; for if s be the length, hanging between P and this point, then from equations (2) art. 26

$$\left.\begin{array}{l} p\cos.\alpha+t\cos.a=0 \\ p\cos.\beta+t\cos.b+s=0 \\ \text{also}\quad \cos.^2 a+\cos.^2 b=1 \end{array}\right\}\ldots(13)$$

from which, cos. a and cos. b being eliminated, there results for t the value $t=\sqrt{p^2-2ps\sin.\alpha+s^2}$

$$=\sqrt{p^2\cos.^2\alpha+(p\sin.\alpha-s)^2}\ldots(14).$$

This expression we may simplify and render independent of p; thus, substitute for s the value in equation (10), and we shall thus have for the tension a at the lowest point A,

$$a=p\cos.\alpha=,\text{(equa. 12)},\ \tfrac{1}{2}L\cot.\alpha\ldots(15),$$

consequently, by substitution, the general expression for t is

$$t=\sqrt{\tfrac{1}{4}L^2\cot.^2\alpha+(\tfrac{1}{2}L-s)^2}\ldots(16)$$

and from this we may get the value of cos. a, by means of the first of (13). It thus appears that when a flexible cord, or chain of given length, is suspended from two points, at a given distance from each other, in the same horizontal line, we may always determine its tension and direction at any point.

(30.) The general equations of the curve (4) and (7), as also the expression (14) for the tension at any point, will become simpler in

form, if the origin of the axes be taken at tne lowest point A of of the curve; for confining ourselves to the consideration of the branch A P_n, we may view A and P_n as the two points of suspension of the cord A P_n, in which case $\alpha=0$, and as y, which has heretofore been measured downwards, will now be measured upwards, we must change the sign which it carries in the preceding formulas when we wish to adapt them to this arrangement of the axis. Calling the tension at A, a, we thus have, by equation (4), $s=\sqrt{2ay=y^2}$ . . . . (1), also from equation (7)

$$x=a\log.\left\{\frac{a+y+\sqrt{(2ay+y^2)}}{a}\right\}\ .\ (2);$$

or since by the equation just deduced

$$\sqrt{a^2+s^2}=\sqrt{a^2+2ay+y^2}=a+y;$$

we may put the last equation under the form

$$x=a\log.\left\{\frac{s+\sqrt{a^2+s^2}}{a}\right\}.\ .\ .\ .\ (3).$$

The expression (14) for the tension t at any point will be

$$t=\sqrt{a^2+s^2}\ .\ .\ .\ .\ (4).$$

All these equations involve the unknown tension a, but this may be determined by trial from (3), since we know the values of x and s in one case, viz. $x=\frac{1}{2}$D and $s=\frac{1}{2}$L. By means of this value of s and a, thus determined, we may obtain y, that is the length of A A′, or the distance of the origin A from the middle of P P_n, and knowing thus the position of the axes, the curve may be constructed from its equation (2).

We shall now give an example of the preceding formulas.

PROBLEM I.—(31.) The length of a heavy flexible chain is just double the horizontal line, joining the points of suspension to determine the pressure on these points, the inclination α to the horizon, &c.

It is obvious, from equation (11, p. 40), that it is sufficient to know the ratio of the length L to the distance D, in order to determine the angle α. We shall not, however, employ this formula, but that above marked (3), as this is of more easy application. If then, in this formula, we suppose s equal to half the length of the chain equal to 1, then x, which is half the horizontal distance between the points of suspension, must be $\frac{1}{2}$; moreover a will then be cot. α (equa. 15, p. 40): hence we shall have

$\frac{1}{2}=a\log.\left\{\frac{1+\sqrt{(a^2+1)}}{a}\right\}$; the logarithm here indicated being hyperbolic, it will be convenient to convert it into a common loga-

rithm which requires that we multiply its value by ·43429, so that $·21715=a \log. \{\frac{1+\sqrt{(a^2+1)}}{a}\}$. Now the first side being little more than $\frac{1}{5}$, a near value of a at once presents itself, viz. $a=\frac{1}{5}=·2$, which substituted in the second member gives $·2 \log. 10·099=·20086$; a result which is rather too small; let us, therefore, take a a little larger, making it $a=\frac{1}{4}=·25$; the second member will then be $\frac{1}{4} \log. 8·1231=·22743$, which is a little greater than the true result. Hence by the known rule of trial and error, since the differences of the results are nearly as the differences of the suppositions which have led to them, we have

$$·22743=·20086=·02657 : ·22743—·21715=·01028 :: ·05 : ·0194$$
$$\therefore a=·25—·0194=·2306 \text{ nearly.}$$

To obtain a still nearer approximation to the truth, let us now assume $a=·23$, then the second member of the equation in a is $·23, \log. 8·8092=·21735$, and as this is a little too great, let us, finally, put $a=·22$ and we have $·22 \log. 9·1996=·21203$; consequently, $·21735—·21203=·00532 : 21735—·21715$

$$=·0002 :: ·01 : ·00037, \therefore a=·23—·00037=·22963$$

Seeking now in the tables for the natural cotangent corresponding to this number, we find for the angle α, or the inclination of the chain to the horizon at either point of suspension, the value $\alpha=77°, 4'$, therefore putting $l=\frac{1}{2}$ L, the pressure on these points is (equation 12), $p=\frac{l}{\sin. \alpha}=\frac{l}{·97463}$; and the tension at the lowest point A is (equation 15) $a=l \cot. \alpha=·22963l$

Also for the distance of A, below the horizontal line P P_n, we have (equation 8), $y=p\ (1 — \cos. \alpha)=·79638l$. the depth of the lowest or middle point.

PROBLEM II.—A heavy flexible chain, 100 feet in length and weighing 1000 lbs. is suspended at its extremities to two fixed points, in the same horizontal line, 95 feet $1\frac{1}{2}$ in. asunder. It is required to determine the greatest depth of the curve, the tension at the lowest part and the tensions at the points of suspension.

In this example the ratio $\frac{D}{L}$ is ·95125; hence, as in the former problem, we shall have to determine, a, or rather cot. α, from the equation

$$·43429\times·95125=·41312=a \log. \{\frac{1+\sqrt{a^2+1}}{a}\}.$$

After a little consideration we find that $a=\frac{7}{4}=1·75$ is a near value, substituting this, therefore, in the last member, we have $\frac{7}{4} \log. 1·7232=·41359$; this being a little greater than the true re-

sult, let us take $x=1\cdot7$, and we then have $1\cdot7$ log. $1\cdot7484=\cdot41249$; consequently,

$\cdot41359-\cdot41249=\cdot0011 : 41359-41312=\cdot00047 :: \cdot05 : \cdot02136$

$\therefore a=1\cdot75-\cdot02136=1\cdot72864$ nearly.

Again, let $a=1\cdot73$, then $1\cdot73$ log. $1\cdot7331=\cdot41316$.

Comparing this result with that obtained by the first supposition, we have the proportion

$\cdot41359-\cdot41316=\cdot00043 : \cdot41359-\cdot41312=\cdot00047::\cdot02:\cdot02219$

$\therefore a=1\cdot75-\cdot02219=1\cdot72781$,

this number corresponds to the natural cotangent of 30°, 4′, therefore, for the inclination α we have $\alpha=30°, 4'$

$$\therefore \text{pressure, } p=\frac{l}{\sin.\alpha}=\frac{50}{\cdot50101}=99\cdot8 \text{ ft.}= 998 \text{ lbs.*}$$

also tension at A, $a=l$ cot. $\alpha=50\times1\cdot72781=86\cdot4$ ft. $=864$ lbs. and distance of A from PP_n, $y=p\,(1-\cos.\alpha)=13\cdot4$ ft.

Problem III.—A chain of given length, $2\,l$ hangs freely over two given points, in a horizontal line, in what position will it rest? (fig. 21).

When the chain is at rest it is plain that what in the former problems was the pressure upon the points of suspension P, P_n, will here be equivalent to the weight of either PP′ or of $P_n P_{n'}$ the parts hanging vertically, these parts are, therefore, equal to each other, and to what we have hitherto called p; hence, calling half the length of the catenary s, the expression for l will be (equa. 10, p. 40,) $l=s+p=p\,(\sin.\alpha+1)$ . . . (1); also the expression for half PP_n or l' is (equa. 9), $l'=p\cos.\alpha\log.\dfrac{\cos.\alpha}{1-\sin.\alpha}$ . . . . (2); dividing this by the last equation we have

$$\frac{l'}{l}=\frac{\cos.\alpha}{1+\sin.\alpha}\log.\frac{\cos.\alpha}{1-\sin.\alpha}=\frac{\sin.\beta}{1+\cos.\beta}\log.\frac{\sin.\beta}{1-\cos.\beta}$$

$$=\tan.\tfrac{1}{2}\beta\log.\frac{1}{\tan.\frac{1}{2}\beta}\dagger=-\tan.\tfrac{1}{2}\beta\log.\tan.\tfrac{1}{2}\beta;$$

the first member of this equation being given, it follows that to determine β, the angle at which the chain is inclined to the vertical, we have only to find by trial a number such that when multiplied by its logarithm, the product shall be equal to a given number.

* Because the weight of a foot of the chain is 10 lbs.

† For $\dfrac{\sin.\beta}{1+\cos.\beta}=\dfrac{\sqrt{1-\cos.^2\beta}}{1+\cos.\beta}=\sqrt{\dfrac{1-\cos.\beta}{1+\cos.\beta}}=\tan.\tfrac{1}{2}\beta$

(*Young's Trigonometry*, p. 37.) In like manner,

$$\frac{\sin.\beta}{1-\cos.\beta}=\sqrt{\frac{1+\cos.\beta}{1+\cos.\beta}}=\frac{1}{\tan.\frac{1}{2}\beta}$$

When this is found, α becomes known, and from the above equation (1), we get $p=\frac{l}{\sin.\alpha+1}$, which gives the length of that part of the chain which hangs vertically on each side of the curve.

PROBLEM IV.—Given the distance $2l'$ between two fixed points in the same horizontal line, to determine the length of the shortest chain that can remain suspended, as in the preceding problem.

We have just seen that $\frac{l'}{l}=-\tan.\frac{1}{2}\beta \log.\tan.\frac{1}{2}\beta$; and as l is to be a minimum, $\frac{l'}{l}$ must be a maximum, l' being constant, that is to say, calling $\tan.\frac{1}{2}\beta$, x, $-x\log.x=-\log.x^x=\log.\frac{1}{x^x}=max.\therefore\frac{1}{x^x}$ max, or $x^x=min.$

This equation is solved at p. 74 of the Differential Calculus, where the value of x is found to be $x=\frac{1}{e}=\frac{1}{2\cdot7182818...}=\tan.\frac{1}{2}\beta$ $\therefore \text{cotan}.\frac{1}{2}\beta=2\cdot7182818...$, $\therefore \frac{1}{2}\beta=20°, 12' \therefore \beta=40°, 24'$. It appears, therefore, that in this case we must have

$$\frac{l'}{l}=-\frac{1}{e}\log.\frac{1}{e}=\frac{1}{e}\ \therefore\ l=el'=2\cdot7182818l',$$

so that if the distance $2l'$ between the fixed points be 10 feet, then the length $2l$ of the shortest chain, which will suspend itself by hanging over them, will be 27·182818 feet.

For the length of each part of the chain hanging vertically we have

$$p=\frac{l}{\sin.\alpha+1}=\frac{l}{\cos.\beta+1}=\frac{l}{2\cos.^2\frac{1}{2}\beta}=\frac{1}{2}l(1+\tan.^2\frac{1}{2}\beta)$$

$=\frac{1}{2}l(1+\frac{1}{e^2})$, and for the distance of the lowest point in the catenary from the horizontal line

$$y=p(1-\cos.\alpha)=p-l\frac{\cos.\alpha}{1+\sin.\alpha}=p-l\tan.\frac{1}{2}\beta$$

$$=\frac{1}{2}l(1+\frac{1}{e^2})-\frac{l}{e}=\frac{1}{2}l\{1-\frac{1}{e}\}^2.$$

For further particulars respecting the curves formed by flexible lines, acted on by different forces, as also respecting those which elastic laminæ assume under like influences, we must refer the student to *Professor Whewell's* Mechanics, chap. x. and xi., (the first edition of this work is here referred to,) where these matters are very elaborately treated, and at great length.

CHAPTER IV.

ON THE EQUILIBRIUM OF A POINT ON A CURVE OR SURFACE.

(32.) If a material point be placed upon a curve surface, and be kept in that place by the mutual action of any number of forces applied to it, the resultant of these forces must be in the direction of the normal to the surface, and must be equivalent to the pressure which the surface sustains. For, if the resultant had any other direction, we might decompose it into two, one in the direction of the normal, and the other in the direction of a tangent to the surface; the first of these would be opposed by the resistance of the surface, but the second, being unopposed, would cause the point to move. Considering, therefore, the resistance which the surface opposes to the normal force, as one of the system of forces acting upon the proposed point, we may altogether disregard the surface and view the point as a free point kept in equilibrium by the system of forces P, P_1, P_2, &c. and N; and, hence, we have the same equation of condition as in (22), that is putting θ, θ', θ'', for the angles which N forms with three rectangular axes, we have

$$\text{N} \cos. \theta + \text{P} \cos. \alpha + \text{P}_1 \cos. \alpha_1 + \text{P}_2 \cos. \alpha_2 + \&c.=0$$
$$\text{N} \cos. \theta' + \text{P} \cos. \beta + \text{P}_1 \cos. \beta_1 + \text{P}_2 \cos. \beta_2 + \&c.=0$$
$$\text{N} \cos. \theta'' + \text{P} \cos. \gamma + \text{P}_1 \cos. \gamma_1 + \text{P}_2 \cos. \gamma_2 + \&c.=0;$$

or, putting as at (22) X, Y, and Z, for the sums of the components along the respective axes, the three equations may be written

$$\text{N} \cos. \theta + \text{X}=0, \ \text{N} \cos. \theta' + \text{Y}=0, \ \text{N} \cos. \theta''+\text{Z}=0 \ . \ (1)$$

Now we must here remark that the angles θ, θ', θ'', which determine the direction of the force N, are entirely dependent on the equation of the surface, and on the co-ordinates of the point to which the forces P, P_1, &c. are applied. Knowing, therefore, the equation of the surface, and the position of the point, we may always determine the direction in which the resultant of the applied forces P, P_1, &c. must necessarily act to ensure the equilibrium. Thus, the equation of the surface being $u=\text{F}(x, y, z,)=0$, we have (*Dif. Calc.* p. 166,) $\cos. \theta=v\dfrac{du}{dx}$, $\cos. \theta'=v\dfrac{du}{dy}$, $\cos. \theta''=v\dfrac{du}{dz}$,

where
$$v=\frac{1}{\sqrt{\dfrac{du^2}{dx^2}+\dfrac{du^2}{dy^2}+\dfrac{du^2}{dz^2}}}.$$

The values of $\cos. \theta$, $\cos. \theta'$, $\cos. \theta''$, being thus found, if we substitute them in the equations (1) and then eliminate N between each two, we shall obtain two equations expressing the conditions which must exist among the applied forces and their inclinations to the

axes, in order that their resultant may be a normal force. Indeed, to obtain these equations, we need not take the trouble to first calculate v, but may simply substitute $\frac{du}{dx}$, $\frac{du}{dy}$, and $\frac{du}{dz}$, for $\theta, \theta', \theta''$, respectively, because v disappears with N. It hence appears, that by means of the equation of the surface, and the position of the point, the equations of equilibrium, originally three, become reduced to two. If, indeed, one of the axes of reference, as the axis of z, coincide with the normal, and, consequently, originate at the point, then to find these two equations it will not be necessary to know the equation of the surface; for then the equations (1) will be

$$\mathrm{P}\cos.\alpha+\mathrm{P}_1\cos.\alpha_1+\mathrm{P}_2\cos.\alpha_2+\&c.=0$$
$$\mathrm{P}\cos.\beta+\mathrm{P}_1\cos.\beta_1+\mathrm{P}_2\cos.\beta_2+\&c.=0$$
$$\mathrm{N}+\mathrm{P}\cos.\gamma+\mathrm{P}_1\cos.\gamma_1+\mathrm{P}_2\cos.\gamma_2+\&c.=0:$$

the two first of which are all that are requisite to establish the equilibrium, for if these hold, the third must necessarily hold, since the intensity of the normal forces or pressure may be any whatever, without disturbing the equilibrium. This third equation is necessary, however, to determine the resultant of the applied forces, or the intensity N of the whole normal pressure.

(33.) Having thus briefly noticed the conditions necessary for the equilibrium of a point on a surface, viewing the matter in the utmost generality, we shall now consider the equilibrium under more particular circumstances, taking those cases only which are likely to present themselves in nature, where the acting forces are gravity or weight.

In this point of view, where we are to consider the means of retaining a heavy point on a given surface, the problem becomes very much simplified from the following considerations. A heavy body (considered as a point) placed upon a surface, will, unless it presses entirely in the direction of the normal, tend to move on that surface towards the horizon, and this tendency will be in a certain determinate direction, viz. in that direction which is nearest the perpendicular to the horizon.

Now if through the point at which the body is placed a tangent plane to the surface be drawn, and from the same point a tangent line, perpendicular to the horizontal trace of the tangent plane, this tangent line will, obviously, be shorter than any other drawn from the same point to the same trace; hence this tangent line must be more nearly perpendicular to the horizon than any other through the body's plane, and, consequently, in the direction of this line the body will tend to move; the very same conditions, therefore, which would be necessary to counteract the tendency to move on the surface, would be necessary to counteract the tendency to move

if the point were placed on this line, instead of on the surface; we may, therefore, in seeking the conditions of equilibrium, substitute the straight line of which we are speaking, for the curve surface. The position of this line is always determinable from the equation of the surface, for the trace is found by putting in the equation of the tangent plane, $z=0$, supposing the plane of xy to coincide with the horizon, and the line sought will be represented by the equations which characterize the perpendicular to this trace through the given point. In the vertical plane through this line must the forces act, so that we may resolve each into two, one acting in this line, and the other in a line perpendicular to it; the sum of the forces acting in this latter line, be it what it may, will be counteracted by the resistance of the line, so that to establish the equilibrium it will merely be necessary that the sum of the forces, acting in the inclined line, be 0.

Hence we shall need but one equation of condition; and, indeed, in whatever two rectangular directions the forces be resolved, the conditions of equilibrium will always be expressed in a single equation. For calling the resistance of the line R, and the sum of the components of the other forces X and Y, we know that the conditions of equilibrium are

$$\left\{\begin{array}{l} X+R\cos.\alpha=0 \\ Y+R\sin.\alpha=0 \end{array}\right\} \therefore \left\{\begin{array}{l} \dfrac{X}{\cos.\alpha}=-R \\ \dfrac{Y}{\sin.\alpha}=-R \end{array}\right\} \cdot (A.)$$

Now each of these equations exists *separately*, whatever be X, or whatever be Y, because R is always equal, and opposite to the pressure $\dfrac{X}{\cos.\alpha}$ or $\dfrac{Y}{\sin.\alpha}$, whatever this may be; and, as the equilibrium requires that they exist *together*, it is merely necessary that we have $\dfrac{X}{\cos.\alpha}=\dfrac{Y}{\sin.\alpha}$ or $X\tan.\alpha-Y=0 \cdot$ (B.)

If we take the axes of components the one parallel and the other perpendicular to the horizon, then the angle α (fig. 22) will be obtuse, and $\cos.\alpha=-\cos\alpha'=-\sin.i$ and the equation of condition just deduced is, therefore, in this case, $X+Y\tan.i=0 \cdot$ (C); i being the inclination of the line of support to the horizon. It must be remembered that when this equation is satisfied, and we wish to determine the resistance R or the pressure on the line, we must recur to one of the equations (A).

From what has now been said, it appears that the equation (C) expresses the conditions of the equilibrium of a heavy body upon any curve surface, i being the inclination of its tendency to move,

in virtue of its weight, to the horizon; and the horizontal and vertical axes of components being taken in the vertical plane of this tendency.

We shall now proceed to the solution of a few problems.

PROBLEM I.—(34.) Given the inclination i, of the straight line AC, to the horizon AB, and the weight of a heavy body W to determine what weight, P acting in a given direction, W M will be sufficient to sustain W on the line (fig. 23).

Here are three forces acting at W, viz. the weight W in the vertical direction WY, the resistance of the line AC acting in the perpendicular direction WR, and the weight P acting in the direction WM, and all these directions are given; hence, as one of the forces W is given, we have enough to determine the other two.

Let us call the given angle CWM, ϵ, then CWQ being equal to $90+i$, we have $MWQ=90+i+\epsilon$, which call θ, then we have sin. $MWQ=\sin.\theta$, $\sin. MWR=\cos.\epsilon$, $\sin. RWQ=\sin. QWN=\sin. i$, consequently, calling the resistance WR, R,

$$\frac{R}{W}=\frac{\sin.\theta}{\cos.\epsilon}\therefore R=W\frac{\sin.\theta}{\cos.\epsilon}=W\frac{\cos.(i+\epsilon)}{\cos.\epsilon}$$

$$\frac{P}{W}=\frac{\sin. i}{\cos.\epsilon}\therefore P=W\frac{\sin. i}{\cos.\epsilon}.$$

If the power P act along the plane, then $\epsilon=0$, and, consequently, in this case, $R=W\cos. i$, $P=W\sin. i\therefore\frac{P}{R}=\frac{\sin. i}{\cos. i}$.

If the power act in a direction parallel to the horizon then $\epsilon=-i$: hence $R=W\frac{1}{\cos. i}=W\sec. i$, $P=W\frac{\sin. i}{\cos. i}\therefore\frac{P}{R}=\sin. i$.

If the power act in a direction perpendicular to the horizon, then $\theta=180°$, also $\cos.\epsilon=\sin. i$, therefore, $R=0$, $P=W$; that is, there must be no pressure upon the line, and, therefore, the power P must be equal to the whole weight which it sustains. If we suppose the power to act perpendicularly to the line, then $\epsilon=90°$, and $\cos.(i+\epsilon)=\sin. i$, and the general formulas give

$$R=W\frac{\sin. i}{-0}=\infty,\ P=W\frac{\sin. i}{0}=\infty.$$

These equations show that the equilibrium cannot be maintained under these circumstances, unless an infinite pressure is exerted on the line requiring an infinite power P. On reviewing the foregoing results, it appears that the power P, necessary to support a weight W on an inclined plane, will be the least possible when it acts in the direction of this plane: indeed it is plain from the

general expression $P = W\frac{\sin. i}{\cos. \varepsilon}$, that P will be the least possible, i remaining the same, when $\cos. \varepsilon = 1$.

If we had solved this problem by the method of resolution, taking for axes the lines WC, WR, as recommended at the former part of last article, then the single equation of equilibrium of which we have there spoken would have been $P \cos. \varepsilon - W \sin. i = 0$. (1); from which we immediately get the value of P sought, viz.

$$P = W\frac{\sin. i}{\cos. \varepsilon}.$$

To determine the pressure we must employ the equation furnished by the other component forces, viz. those acting in WR, this equation is $R + P \sin. \varepsilon - W \cos. i = 0 \ldots.$ (2.)

or $$R = W\frac{\sin. i \sin. \varepsilon - \cos. i. \cos. \varepsilon}{\cos. \varepsilon} = 0$$

$$\therefore R = W\frac{\cos. i \cos. \varepsilon - \sin. i \sin. \varepsilon}{\cos. \varepsilon} = W\frac{\cos. (i+\varepsilon)}{\cos. \varepsilon}.$$

By employing the second mode of resolving the forces, that is according to horizontal and vertical axes, we should have for the conditions of equilibrium the equation (C), in this case,

$$P \cos. (\varepsilon + i) + \{P \sin. (\varepsilon + i - W\}\frac{\sin. i}{\cos. i} = 0.$$

whence $$P \cos. \varepsilon \cos.^2 i + P \cos. i \sin.^2 i = W \sin. i$$

$$\therefore P \cos. \varepsilon = W \sin. i \therefore P = W\frac{\sin. i}{\cos. \varepsilon}.$$

For the resistance R we must employ one of the equations (A); taking the first we have $P\frac{\cos. (\varepsilon+i)}{\sin. i} = R = W\frac{\cos. (\varepsilon+i)}{\cos. \varepsilon}$ as before.

PROBLEM II.—Given the position of the line AC, and of the pulley M, as also the weights of W and P, to determine whereabouts W must be placed that the equilibrium may be possible (fig. 24).

The perpendicular MM′ is given because the position of M and of AC are given. Call this perpendicular a; then by equation (1); last proposition,

$$\cos. \varepsilon = \frac{W}{P} \sin. i \therefore \sin. \varepsilon = \frac{\sqrt{P^2 - W^2 \sin.^2 i}}{p}.$$

Now $WM \sin. \varepsilon = MM' = a$

$$\therefore WM = \frac{a}{\sin. \varepsilon} = \frac{Pa}{\sin. i \sqrt{P^2 - W^2 \sin.^2 i}};$$

an equation which determines the place of W.

Problem III.—Two weights W, W′, attached to the extremities of a string, which passes over a fixed pully, mutually support each other on two inclined planes (fig. 25), to determine the relations between W, W′, the tension of the string, and the pressures on the planes.

Here each weight is supported by the tension of the string, which is the same throughout; this tension will then be the same, as regards each body, as the power we have hitherto called P.

If, therefore, we designate the angles concerned in one of the planes as in prob. I., we have $\frac{R}{W}=\frac{\cos.(i+\varepsilon)}{\cos.\varepsilon}$, $\frac{P}{W}=\frac{\sin. i}{\cos. \varepsilon}$.

In like manner, for the other plane we have

$$\frac{R'}{W'}=\frac{\cos.(i'+\varepsilon')}{\cos.\varepsilon'},\ \frac{P}{W'}=\frac{\sin. i'}{\cos.\varepsilon'}\ \therefore\ \frac{W}{W'}=\frac{\sin. i' \cos.\varepsilon}{\sin. i \cos.\varepsilon'};$$

which equations exhibit the relations required.

If the pulley be fixed at the intersection of the planes, so that the string acts in each plane, then $\varepsilon=0$, $\varepsilon'=0$ and $\frac{W}{W'}=\frac{\sin. i'}{\sin. i}=\frac{CA}{CA'}$; that is, the weights are in this case as the lines on which they rest.

It has been already observed that when the weight rests on a point of a curve, the conditions are the same as if it rested on the tangent line through that point; the inclination of this line to the horizon, which it is necessary to know, may be determined when we know the equation of the curve, referred to vertical and horizontal axes, and the co-ordinates of the point where the body is placed. If we resolve all the forces which are applied to the point, in the directions of the axes, and call the inclination of the tangent line to the axis of x, i, then we know that the conditions of equilibrium will be expressed by the single equation (C), at page 47, viz. $X+Y \tan. i=0$; but, if x, y are the co-ordinates of the point, we know (*Diff. Calc.* p. 113,) that $\tan. i=\frac{dy}{dx}$, hence the equation of condition is $X+Y\frac{dy}{dx}=0 \ldots$ (1.)

If a weight W be supported on a curve by means of another weight P (fig. 26), hanging vertically, the two weights being connected by a flexible string passing over a pulley, then it will be most convenient to take the vertical line MX as axis of x, and the horizontal line MY as axis of y. In this case we shall have the following values for the forces X and Y, viz. for X we shall have the weight W diminished by P cos. MWn=P cos. WMm, and for Y we shall have $-$P cos. MWm, that is, putting MW$=r$,

$$X = W - P\frac{x}{r},\ Y = -P\frac{y}{r};$$

substituting these values in the above equation, we have

$$W - P\frac{x}{r} - P\frac{y}{r}\cdot\frac{dy}{dx} = W - \frac{p}{r}\{x + y\frac{dy}{dx}\} = 0;$$

but, since $r^2 = x^2 + y^2 \therefore r\frac{dr}{dx} = x + y\frac{dy}{dx}$; hence the equation of equilibrium is $W - P\frac{dr}{dx} = 0 \ldots . (2)$.

Problem IV.—A given weight W rests upon a circular arc, as in fig. 26, being supported by another given weight P, by means of a string passing over a pulley, fixed at a given point in the vertical line MX, passing through the centre C: to determine the position of W.

Referring the curve to the vertical and horizontal axes MX, MY, and calling MC, x', we have, for the equation of the curve, $x'^2 - 2x'x + x^2 + y^2 = r^2$; for substituting l, the length of the string MW, for $x^2 + y^2$, $l^2 - 2x'x = r^2 - x'^2$. Hence differentiating with respect to x, $l\frac{dl}{dx} - x' = 0 \therefore \frac{dl}{dx} = \frac{x'}{l}$, so that the general equation of equilibrium (2) is, in this case,

$$W - P\frac{x'}{l} = 0 \therefore l = \frac{Px'}{W};$$

which gives the position of W. Or, because

$$l^2 = x^2 + y^2 = r^2 + 2x'x - x'^2, \therefore x = Mm = \{\frac{P^2}{W^2} + 1\}\frac{x'}{2} - \frac{r^2}{2x'}.$$

Problem V.—Instead of a circle let the curve of support be an hyperbola with its transverse diameter vertical, the pulley being in the centre (fig. 27).

The equation of the curve is $a^2y^2 - b^2x^2 = -a^2b^2$; and the expression for l, the distance of any point in it from the centre, is (*Anal. Geom.* p. 143-4,)

$$l^2 = \frac{a^2 + b^2}{a^2}x^2 - b^2 = e^2x^2 - b^2;\ e^2 \text{ being put for } \frac{a^2 + b^2}{a^2}.$$

Hence, by differentiating, $l\frac{dl}{dx} = e^2x \ \therefore\ \frac{dl}{dx} = \frac{e^2x}{l}$,

and the equation of equilibrium is, therefore,

$$W - P\frac{e^2x}{l} = 0,\ \therefore \frac{x}{l} = \cos. \angle M = \frac{W}{Pe^2}.$$

This fixes the position of W; or if we substitute for l its value in terms of e and x, as given above, we shall have

$$x^2=\frac{W^2}{P^2e^2}x^2-\frac{W^2b^2}{P^2e^4}. \quad \therefore x=\frac{Wb}{e\sqrt{W^2-P^2e^2}}=Mm.$$

It appears from this expression that the equilibrium is impossible if W is less than Pe; and if $W=Pe$ the point of rest must be at an infinite distance.

PROBLEM VI.—It is required to find a curve such that a given weight P hanging over the pulley may balance another given weight W at every point of it (fig. 28).

We have here to find a curve such that the equation

$$W-P\frac{dr}{dx}=0; \text{ or } W-P\frac{d(x^2+y^2)^{\frac{1}{2}}}{dx}=0$$

may exist not only at one particular point, as in the preceding cases, but at every point (x, y) of the curve. This equation, therefore, can be no other than the differential equation of the sought curve. Hence, multiplying by dx and integrating, there results

$$Wx-Pr+C=0, \text{ or } Wx-P\sqrt{x^2+y^2}+C=0, \text{ or}$$

$$y^2+\frac{P^2-W^2}{P^2}x^2-\frac{2WC}{P^2}x-\frac{C^2}{P^2}=0 \ \ldots\ldots (1);$$

for the equation of the curve sought. In order to simplify this equation, let us remove the term containing the first power of x, which is done by substituting for x, in this equation, the value (*See Anal. Geom.* p.173-4,) $x=\frac{WC}{P^2-W^2}+X$; which leads to the equation, $Y^2-\frac{W^2-P^2}{P^2}X^2=-\frac{C^2}{W^2-P^2}$ (2); and this equation characterizes an hyperbola related to its principal axes. For the distance c between the centre and focus of this hyperbola, we have (*Anal. Geom.*, p. 171,) $c=\frac{WC}{W^2-P^2}$; but this is the distance of the new origin from the primitive origin, and the primitive origin is on the pulley; hence the pulley is at the focus of the hyperbola.

By putting first $Y=0$ and then $X=0$ in the equation (2), we have for the semi-axes of the hyperbola,

$$A=\frac{PC}{W^2-P^2},\ B=\frac{C}{(W^2-P^2)^{\frac{1}{2}}};$$

in which equations C is arbitrary.

SECTION II.

ON THE EQUILIBRIUM OF A SOLID BODY.

(35.) Having considered pretty much at large the equilibrium of forces, acting upon a free point, it is time now to examine the more general case in which forces act upon different points, all connected together in an invariable manner, as we shall here suppose the parts of a solid body to be. We shall divide the theory into two parts; first, considering the forces which act upon the body to be all parallel, and then considering them to act in any manner whatever.

CHAPTER I.

ON PARALLEL FORCES

(36.) Let us first consider two parallel forces P, P_1, acting at the extremities of a straight line AB, (fig. 29,) and let it be required to determine what must be the intensity, and where the point of application, of a single force, which, acting on the line, shall have the same effect as these two. Let us represent the parallel forces by the lines AP, BP_1, and let us apply to the extremities of the line any two equal but opposite forces AM, BM_1; these will destroy each other, and will, therefore, have no effect on the system. Hence, instead of the two forces AP, BP_1, acting on the line, we may consider as acting the four forces AM, AP, BM_1, BP_1, or the resultants of these, AR, BR_1. We have thus exchanged our two parallel forces for two oblique forces, meeting in some point C. Considering the lines to be all rigid, we may transfer the points of application of these forces to their point of concurrence C, making $CE = AR$, and $CE_1 = BR_1$, so that these concurring forces, acting on C, have the same effect on the rigid line AB, with which they are connected by the rigid lines CA, CB, as the original forces P, P_1. Let us now resolve the forces CE, CE_1, into their original components Cm, Cp, and Cm_1, Cp_1; then since the two Cm, Cm_1, are equal and opposite they destroy each other, so that the system will be reduced to the two conspiring forces Cp, Cp_1, or to the single force $Cp+Cp_1$, which is equal to $AP+BP_1$; thus we have, for the intensity of the resultant of the two forces P, P_1, $P+P_1 = R$; and as this force may be applied at any point, in its direction

CO, O will be that point in the proposed line to which it must be applied; the situation of this point is thus determined. By similar triangles,

$$\frac{CO}{AO}=\frac{Cp}{Ep} \text{ and } \frac{OB}{CO}=\frac{Ep_1}{Cp_1} \therefore \frac{OB}{AO}=\frac{Cp.Ep_1}{Ep.Cp_1}=\frac{Cp}{Cp_1};$$

that is OB : AO : : Cp : Cp_1, or Cp, Cp_1 being equal to P, P_1; OB : AO : : P : P_1.

Hence we conclude that *the resultant of two parallel forces is also parallel, is equal to their sum, and acts at that point which divides the distance between them into parts reciprocally proportional to their intensities*. This point therefore is fixed however the *direction* of the components may vary.

The proportion, just deduced, gives also (see fig. 30,)

$$AB : OB :: R : P$$
$$AB : AO :: R : P_1;$$

therefore P, P_1, and R are to each other, respectively, as OB, OA, AB, that is to say, that *any two of the three forces are to each other reciprocally as their distances from the third*, so that when any three of the six quantities concerned, viz. the three forces and the three distances are given, the other three may be determined by the successive application of this theorem. The same theorem then serves to divide a given force R into two others parallel to it, acting at given distances OA, OB, on each side of O. And lastly, it serves also to determine the intensity and point of application of that force P_1 (fig. 31) which will keep in equilibrium the line AO, acted upon by two opposing parallel forces P, R′, whenever such equilibrium is possible. This qualification is necessary, because there is one case in which two parallel forces, acting on opposite sides of a line, cannot be equilibrated by any third force, viz. the case in which the two forces are equal; for it is plain that if R′ be equal to P, that R, which is equal and opposite to R′, cannot be the resultant of P, and any other force P_1, because if it were we should have $P+P_1=R$, whereas P alone is equal to R: hence P_1 must be 0, and therefore, by the theorem, the point of application B must be infinitely distant, so that no single force can keep the line AO at rest when its extremities are solicited by equal parallel forces acting in opposite directions. The tendency of these forces will plainly be to cause the line to turn about its middle point, this being at rest.

From what has now been said it follows that *the resultant of two opposite forces*, P, R′, *applied to different points*, A, O, *is equal to their difference* P_1', *acting parallel to them in the direction of the greater, and that its point of application* B *is given by the proportion*.

$$P'_1 : R' :: AO : AB \therefore AB=\frac{R'}{P'_1}AO=\frac{R'}{R'-P}AO.$$

From knowing how to compound two parallel forces acting upon a straight line, we are enabled to compound any number so acting. The resultant will, obviously, be equal to the algebraic sum of the components, affixing opposite signs to those which draw in opposite directions. As to the point of application of this resultant we shall not stop to determine it for this particular case, but shall proceed to consider the theory of parallel forces in all its generality.

(37.) Let P, P_1, P_2, &c. be parallel forces, applied to any system of points, A, A_1, A_2, &c. any how situated in space, but invariably connected by rigid lines, (fig. 32,) and let it be required to determine the resultant of this system both in intensity and position.

The most obvious mode of proceeding is this, viz. first to compound two of the forces P, P_1, and to substitute for them their resultant R; then to compound this with the third force P_2, and to substitute for the two R, P_2, that is, for the three P, P_1, P_2, their resultant R_1, and so on till the system is reduced to the two parallel forces $R_n - 1$, P_n, of which the resultant will be that of the whole system; and will, therefore, be equal in intensity to the sum of the components. In this process of composition the several partial resultants R, R_1, R_2, &c. are not only determined in intensity, but the point of application in the line joining the points, acted on by the two components, is in each case determined. Every such point would remain fixed, however the direction of the component parallel forces might vary, provided their respective intensities did not vary (36); and, therefore, the point of application of the final resultant would remain fixed, however the direction of the system of parallel forces might vary provided they retained their respective intensities; it is through this point, therefore, that the resultant of the system must always pass under every change of direction; it is hence called the centre of these parallel forces.

(38.) Let now there be assumed any three rectangular axes, and let us represent by

$x, y, z,$ the co-ordinates of the point A
$x_1, y_1, z_1,$ . . . . . A_1,
$x_2, y_2, z_2,$ . . . . . A_2,
&c. &c.

and by X, Y, Z, those of the centre of the parallel forces; we shall prove that these latter are severally

$$\left.\begin{aligned} X &= \frac{Px + P_1x_1 + P_2x_2 \ldots}{P + P_1 + P_2 \ldots\ldots} \\ Y &= \frac{Py + P_1y_1 + P_2y_2 \ldots}{P + P_1 + P_2 \ldots\ldots} \\ Z &= \frac{Pz + P_1z_1 + P_2z_2 \ldots}{P + P_1 + P_2 \ldots\ldots} \end{aligned}\right\} \ldots (1).$$

For let us first consider only two forces P, P_1, acting on the points A, A_1 (fig. 33), of which the abscissas are $OA'=x$ $AO_1'=x_1$, and let C be the centre of these two forces, its abscissa being $OC^{1\prime}=X'$; then if a, c, a_1, be the projections of A, C, A_1, on the plane of xy, we shall have, on account of the parallels,

$$A_1C : AO :: a_1c : ac :: A'_1C'=x_1-X' : A'C=X'-x;$$

but (36), $A_1C : AC :: P : P_1 \therefore x_1-X' : X'-x :: P : P_1$

$$\therefore (P+P_1)\,X'=Px+P_1x_1 \therefore X'=\frac{Px+P_1x_1}{P+P_1}.$$

Let us now proceed with the two forces $(P+P_1)$ and P_2, after having joined their points of application C, A_2, exactly as we have proceeded with P and P_1, and, calling the abscissa of the centre of our new forces X'', the result must be

$$(P+P_1+P_2)X''=(P+P_1)\,X'+P_2x_2\,;$$

or, substituting for X' the value just obtained

$$(P+P_1+P_2)\,X''=Px+P_1x_1+P_2x_2$$

$$\therefore X''=\frac{Px+P_1x_1+P_2x_2}{P+P_1+P_2}.$$

Proceeding in this manner till we arrive at the centre of all the parallel forces, of which the abscissa is X, we shall have, finally,

$$X=\frac{Px+P_1x_1+P_2x_2\ldots}{P+P_1+P_2\ldots}$$

as announced; and if for the axis of x we substitute successively the axes of y and of z, we shall have the similar equations

$$Y=\frac{Py+P_1y_1+P_2y_2\ldots}{P+P_1+P_2\ldots}$$

$$Z=\frac{Pz+P_1z_1+P_2z_2\ldots}{P+P_1+P_2\ldots};$$

and thus we may always determine the co-ordinates of the centre when we know those of the points of application of the system of parallel forces, as well as the several intensities of those forces. Calling the resultant of the forces R, the preceding equations give

$$\left.\begin{aligned}RX&=Px+P_1\,x_1+P_2x_2\,\cdot\\RY&=Py+P_1\,y_1+P_2y_2\,\cdot\\RZ&=Pz+P_1\,z_1+P_2z_2\,\cdot\end{aligned}\right\}\cdot(2).$$

We may here remark that it is possible so to place the axes of co-ordinates, that two of the three equations (2) will suffice to fix the position of the resultant of the system; for let one of the axes, as the axis of z, be taken parallel to the direction of the forces, then, as the resultant itself will be parallel to the same axis, its position will be known if we only know where it meets the plane of xy, that is, if we know the X, Y, of any point in it; hence the two first of equations (2) are sufficient to determine the line in which the resultant acts, and this is all we want to know, since on

whatever point in this line it acts, the effect is the same. Under this arrangement of the axes, therefore, the equations necessary for the determination of the resultant in intensity and position are

$$\left.\begin{aligned} R&=P+P_1+P_2+P_3+\cdot \\ RX&=Px+P_1\,x_1+P_2\,x_2+P_3\,x_3+\cdot \\ RY&=Py+P_1\,y_1+P_2\,y_2+P_3\,y_3+\cdot \end{aligned}\right\}\cdot\cdot\,(3).$$

(38.) The product of any force, by the perpendicular distance of the point on which it acts from any plane, is called the *moment* of that force with respect to the plane; thus Px is the moment of the force P with respect to the plane of YZ, because x is the distance of the point A on which it acts from that plane. Hence we learn from either of the three equations (2) just given, that *the moment of the resultant of a system of parallel forces in reference to any plane is equal to the sum of the moments of the components in reference to the same plane;* the algebraical sum being always understood, regard being had to the signs of the forces as well as to the co-ordinates of the points on which they act. It is easy to see how the foregoing results become abridged when the forces all act in one plane, as also when the several points on which they act are in one straight line; in the former case only one co-ordinate plane is necessary, viz. the plane in which all the points are situated; in the latter case only one axis is necessary, viz. the line in which the points are situated, so that either one or two of the foregoing general equations may in particular cases become superfluous.

The preceding theory will enable us readily to determine the conditions of equilibrium of a system of parallel forces; for let us assume the axis of z parallel to the direction of the forces, then, since the sum of the forces, that is the resultant R, is 0, we have, by the equations marked (3),

$$\left.\begin{aligned} P+P_1+P_2+P_3+\cdot&=0 \\ Px+P_1x_1+P_2x_2+P_3x_3+\cdot&=0 \\ Py+P_1y_1+P_2y_2+P_3y_3+\cdot&=0 \end{aligned}\right\}\cdot\,(4).$$

which are the equations necessary to establish the equilibrium, and they express these conditions, viz.

1st. *The sum of the forces must be equal to* 0.

2d. *The sum of their moments, in reference to each of two perpendicular planes parallel to their direction, must be equal to* 0.

(39.) Before terminating this chapter, we should remark, that a more concise notation is frequently employed to express the equations (1), (2), &c. thus the equations (1) are written

$X=\dfrac{\Sigma\,(Px)}{\Sigma\,(P)}$, $Y=\dfrac{\Sigma\,(Py)}{\Sigma\,(P)}$, $Z=\dfrac{\Sigma\,(Pz)}{\Sigma\,(P)}$, the character Σ signifying the sum of the whole system of quantities of the form of that to which it is prefixed. In like manner the equations (2) may be

written $$RX = \Sigma\,(Px),\ RY = \Sigma\,(Py),\ RZ = \Sigma\,(Pz),$$
and the equations of equilibrium according to this notation are
$$\Sigma\,(P) = 0,\ \Sigma\,(Px) = 0,\ \Sigma\,(Py) = 0,$$
provided the two perpendicular planes to which the moments are referred are parallel to the direction of the forces. We shall now proceed to some interesting and important applications of the theory delivered in this chapter.

CHAPTER II.

ON THE CENTRE OF GRAVITY.

(40.) EXPERIENCE teaches us that all bodies within our reach tend towards the earth, to which, if abandoned to themselves, or left unsupported, they would fall in a vertical direction. The reason why smoke and vapours in general do not fall to the earth, is that they are not left unsupported, being indeed borne up by the air in the same way that a piece of wood is borne up by the water in a vessel, and prevented from reaching the bottom as it would do if this support were removed. This universal tendency of all bodies to the earth, proves the existence of a soliciting power whose influence extends equally to every body with which we are surrounded. By the tendency here spoken of we mean the disposition to move, and that this is the same in all bodies, great and small, when all support is taken away, has been fully and frequently established by the most convincing experiments; thus if a very small particle be placed beside a large mass in a vessel exhausted of air, they will, when let go, continue beside each other during the whole time of descent, and will both strike the bottom of the vessel at the same instant, so that if it were possible to destroy the cohesion among the particles of matter, in virtue of which it becomes a solid mass, thus enabling each particle to obey whatever force acted upon it individually, yet the tendency to move being exactly the same in each, no one particle could in descending displace any other, so that the same arrangement would be preserved during the descent as if all the particles cohered. We call this soliciting power of the earth, which we sée is altogether independent of the mass on which it acts, THE FORCE OF GRAVITY, or simply GRAVITY, thus naming an influence, the nature of which we know nothing only as regards its effects, and this is in fact all that we here require to know of it.

It may be proper here to caution the student against a very common misapplication of the term gravity. We use incorrect language

when we speak of the gravity of this body, or the gravity of that, for gravity is not of the body but of the earth, and is always the same at the same place, exerting the same effect on all bodies however different, that is, producing in all the same tendency to move. The weight of a body furnishes us with no information respecting the force of gravity, but only with respect to the number of its constituent particles, for if one body is double the weight of another this does not arise from any variation in the force of gravity, but because there are twice the number of particles in one body that there are in the other, and each particle is influenced alike; so that it will require double the effort to support one that it requires to support the other; the lighter body may, however, have more external surface or appear under greater bulk than the heavier, but then the pores which separate the component particles will be proportionally larger.

Understanding by the weight of a body the effort necessary to prevent its falling, we may correctly say that the weight of a body is the resultant of all the efforts (or weights) which gravity impresses upon its component particles; as these component efforts are all directed in parallel lines, their resultant must be equal to their sum, and act in their common direction and at a point which will be the centre of these parallel forces, and which in the present case is called the *centre of gravity* of the body. The determination of this centre in different bodies may be effected by the application of the theory delivered in the preceding chapter, provided we suppose, as we shall here do, that the bodies proposed are perfectly homogeneous, so that the effort necessary to counterbalance the influence of gravity on any part of the body will be proportional to the mass of that part.

Before proceeding to particular applications of the theory, we may as well here notice the distinguishing characteristics of the point which we have called the centre of gravity, and which are direct inferences from that theory; these are, 1st, that if the centre of gravity be supported, the whole body will be in equilibrium, because the resultant of all the forces which act on it will be opposed in whatever position the body be placed: moreover every body kept in equilibrium by a single force, must have its centre of gravity in the line of direction of that force. 2d. The sum of the products of each particle of a body into its distance from any plane, the distances on opposite sides taking opposite signs, is equal to the product of the whole mass into the distance of its centre of gravity from the same plane; so that if a plane divide a body into symmetrical halves, it must pass through the centre of gravity. The first of these properties points out an experimental method of finding the centre of gravity of a body: thus, let the body be suspended by a string attached to any point, it will

arrange itself so that this string would, if we could continue it, pass through the centre of gravity. In like manner, if it were suspended from any other point, the line of the string would also pass through the centre of gravity, consequently the intersection of these two lines would determine that centre. If the body have a flat surface, we may lay it on a horizontal table pushing it more and more over the edge till it just balances itself, in which position the centre of gravity will be vertically over the edge of the table; if, then, we mark the line of the edge on the body, and proceed in the same way with the body in another position, we shall thus have a point in the same vertical as the centre of gravity, and to which if, as a support, an indefinitely slender vertical rod were applied, and the table removed, the body would remain in equilibrium. Or if it were to be suspended by this point, the flat surface would assume a horizontal position.

(41.) We shall now investigate general analytical expressions for the determination of the centre of gravity of any body whatever.

Let ABC, &c. (fig. 34,) represent any solid body, the component particles of which we shall call P, P_1, P_2, &c. and their sum or the mass of the whole body, B. Then, if G be the centre of gravity of this body, and the body be referred to three rectangular planes, the distance of G from the plane of zy will, by equation (1), page 55, be $X = GH = \frac{Px + P_1x_1 + P_2x_2 + \dots}{B}$ .... (1).

The numerator of this fraction consists of the sum of the innumerable particles P, P_1, P_2, &c., multiplied by their respective distances from the plane of zy; but although the terms are innumerable, yet their sum may be accurately determined by the aid of the integral calculus. In order to this determination, let CN be any increment of the body, then the corresponding increment of the expression under consideration, that is of the numerator of (1), will be equal to the sum of all the particles in the slice CN, multiplied by their respective distances from the plane of zy. Now, calling the increment MN of the abscissa, h, it is obvious that however small we take h, that is however slender the slice CN may be, the sum of which we have just spoken will always be comprised between these two, viz. the sum of the same particles when multiplied each by the distance $AM = x$, and the sum when multiplied each by $AN = x + h$; that is, putting S for the expression we are considering, and Δ S, Δ B, for the corresponding increments of this and of the body, Δ S will always be intermediate between $x \Delta B$ and $(x+h) \Delta B$, but the ratio of these is

$$\frac{(x+h)\Delta B}{x \Delta B} = 1 \text{ in the limit,}$$

or when h, and consequently Δ B, is 0; therefore the ratio of the intermediate quantity Δ S to either must in the limit be 1; that is,

$$\frac{\Delta S}{x\Delta B}=1 \text{ in the limit, that is, } \frac{dS}{xdB}=1 \therefore dS=xd\,B \therefore S=\int xdB;$$

hence the expression (1) is

$$\left.\begin{array}{l} X=GH=\dfrac{\int xdB}{B} \\ \text{In like manner} \\ Y=\dfrac{\int ydB}{B}, Z=\dfrac{\int zdB}{B} \end{array}\right\} \ldots\ldots (A),$$

equations from which the co-ordinates of the centre of gravity of B may be determined when the equation of B is known.

But it must be observed, that though in all these equations dB signifies the differential of the body, yet it is not to be represented in all by the same analytical expression: for regard must be had to the position of the slice Δ B, as this will in general be different in its three positions, parallel to the rectangular planes; and therefore also, in general, the expressions for dB will all three be different; but this will be shown more clearly when we come to apply the formulas to the determination of the centres of gravity in surfaces and solids, (art. 43.)

It is seldom, however, requisite to employ all three of these equations for that purpose; much will depend upon a happy arrangement of the co-ordinate axes: thus, if we know the position of a plane that will divide the body into symmetrical halves, then we know that the centre of gravity must lie in this plane, (p. 59;) taking, therefore, this for one of the co-ordinate planes, it is clear that two of the equations (A) will suffice to determine the centre. If we know two perpendicular planes, of which each divides the body into symmetrical halves, and in all bodies of revolution any two planes through the axis of revolution will do this, then we also know the line in which the centre lies, and hence one of the above equations will be sufficient. Should the body be merely a lamina of matter, so thin, indeed, as to be taken for a plane surface, then, by choosing the axis in this plane, more than two of the foregoing equations can never be requisite, and but one if one of the co-ordinate axes divide the figure symmetrically into halves; and the same is obviously true if the body be considered merely as a plane line. If the curve be of double curvature, all three of the equations will generally be necessary.

Let us now proceed to the actual determination of the centres of gravity in given figures, considering in order lines, surfaces, and solid bodies.

Determination of the Centre of Gravity of a Plane Line.

(42.) When the body may be considered as a line lying in one plane, we shall put $2s$ for B; and supposing first that the line is symmetrically situated with respect to the axis of x, that is, that the centre is in this axis, we shall have by the first of (A) this expression for the distance of the centre from the origin, viz.

$$X=\frac{\int x ds}{s}=\frac{\int x\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx}{s}.$$

But when the line is not symmetrical with respect to the axis, then we must determine the Y of the centre of gravity as well as the X, and this, by the second of equation (A), is

$$Y=\frac{\int y ds}{s}=\frac{\int y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx^2}{s}, \text{ in which } \frac{dy}{dx}$$

is given in terms of x by the equation of the line.

PROBLEM I.—To determine the centre of gravity of a given straight line.

In this case we have $X=\frac{\int x dx}{x}=\frac{\frac{1}{2}x^2}{x}=\frac{1}{2}x$, therefore, representing the whole line by a, we have, when $x=a$, $X=\frac{1}{2}a$, so that the centre of gravity is at the middle point, as indeed is obvious without calculation.

PROBLEM II.—To determine the centre of gravity of the contour of any polygon.

Let us represent the sides of the polygon by P_1, P_2, P_3, &c. and let the co-ordinates of the angular points be

$$x_1, y_1;\ x_2, y_2;\ x_3, y_3;\ \&c.,$$

then the co-ordinates of the middle points of the sides will be

$$\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2};\ \frac{x_2+x_3}{2}, \frac{y_2+y_3}{2};\ \&c.$$

and these points are, by last problem, the several centres of gravity of the sides, so that we now have to find the centre of gravity of the weights P, P_1, P_2, &c. acting at these points, and for this we have the equations

$$X=\frac{P_1(x_1+x_2)+P_2(x_2+x_3)+P_3(x_3+x_4)+\ldots P_n(x_n+x_1)}{2(P_1+P_2+P_3+\ldots\ldots P_n)}$$

$$Y=\frac{P_1(y_1+y_2)+P_2(y_2+y_3)+P_3(y_3+y_4)+\ldots P_n(y_n+y_1)}{2(P_1+P_2+P_3+\ldots\ldots P_n)}.$$

It is obvious that if the polygon be regular, the middle point, or the centre of the inscribed circle, will be the centre of gravity, and P_1, P_2, P_3, &c. will be all equal; hence if the polygon have n sides

$$X=\frac{2\,(x_1+x_2+x_3+\,..\,x_n)}{2n}$$

$$\therefore nX=x_1+x_2+x_3+\,...\,x_n,$$

an equation which expresses this geometrical property, viz. that if, from the corners, and from the centre of a regular polygon, perpendiculars to any line in its plane be drawn, the sum of the perpendiculars from the corners will be equal to as many times that from the centre as there are sides to the polygon.

Problem III.—To determine the centre of gravity of a circular arc $BAC=2s$ (fig. 35). The equation of the curve referred to the axes AX, AY is

$$y^2=2rx-x^2$$

$$\therefore \frac{dy}{dx}=\frac{r-x}{\sqrt{2rx-x^2}}\;\therefore \sqrt{1+\frac{dy^2}{dx^2}}=\frac{r}{\sqrt{2rx-x^2}}$$

$$\therefore X=\frac{r}{s}\int\frac{x\,dx}{\sqrt{2rx-x^2}}=\frac{r}{s}\{-\sqrt{2rx-x^2}+s\}\ (\textit{Int.Calc.}\ \text{p.92-3}).$$

$$=\frac{r}{s}(s-y)=r-\frac{ry}{s}\therefore OG=r-X=\frac{ry}{s},$$

so that the distance of the centre of gravity from the centre of the circle is a fourth, proportional to the arc, the radius, and the chord of the arc.

When the arc is a semicircle the chord is double the radius, and then

$$OG=\frac{2r}{3\,.\,141593\,..}=\frac{r}{1\,.\,57079}=\cdot 63662r,$$

and when it is a whole circle, then, y being $=0$, OG is 0, as we otherwise know to be the case.

It may be here observed, that in this solution we have not sought for the arbitrary constant necessary to complete the above integral, nor need we seek for it in any case, because it is always the definite integral that we want. Since the body whose centre of gravity we require is necessarily limited, it is between its limits that our integral is to be taken, and thus limited it can require no correction, (*Int. Calc.* p. 90-1.)

Problem IV.—To determine the centre of gravity of the arc of a cycloid.

The differential equation of this curve is, (*see Int. Calc.* p. 115,)

$$\frac{dy}{dx}=\sqrt{\frac{2r-y}{y}},$$

where r is the radius of the generating circle

$$\therefore \sqrt{1+\frac{dx^2}{dy^2}}=\sqrt{\frac{2r}{y}} \therefore s=\sqrt{2r}\int y^{-\frac{1}{2}}dy=2\sqrt{2ry}$$

$$\therefore Y=\frac{1}{s}\int yds=\frac{1}{2\sqrt{\{2ry\}}}\sqrt{\{2r\}}\int y^{\frac{1}{2}}dy=\frac{1}{3}y.$$

For the whole curve $y=2r$, so that the distance of the centre of gravity G from the vertex is equal to one third of the diameter of the generating circle.

Determination of the Centre of Gravity of a Plane Area.

(43.) When the body is considered as a plane area u, the first of the equations (A) becomes, by substituting u for B,

$$X=\frac{\int xdu}{u}=\frac{\int xydx}{\int ydx} \dots (1),$$

which is of itself sufficient to determine the centre when the axis passes through it, that is when this axis divides the figure into symmetrical halves. But suppose we require the centre of gravity of the part u on one side the axis, that is of the area ABD (fig. 36), then, in addition to X=AM, we must also know Y=MG.

Now P being any point (x, y), we are not to substitute in the expression for Y, instead of dB, the value ydx, which we put in the expression for X; because now we want, agreeably to the general investigation, not the differential of APN, but that of APN′D; the former was correctly expressed by PN . dx, but the latter must be PN′ . dy=(AD—AN) $dy=(a-x)\,dy$, a being the distance of the bounding ordinate from the origin; hence

$$Y=\frac{\int ydu}{u}=\frac{\int(a-x)\,ydy}{\int(a-x)\,dy} \dots (2).$$

It appears then that the expression for du depends upon the manner in which we conceive u to be generated, that is whether we consider the increment Δu to be paralel to the axis of y or to the axis of x, or, which is the same thing, whether we consider in the equation of u the independent variable to be x or y. We may, however, express du in a form which will leave it optional with us which shall be the independent variable; this form is $du=\int dx\,dy$, where the integral sign applies to *either* of the differentials under it, regard being had in the integration to the limits between which the variable is comprised. Thus, in the case we are considering, if we take the integral sign to apply to dx, and integrate between the limits $x=a$ and $x=x$, then the above equation is the same as $du=(a-x)\,dy$; but if we consider the integral sign to apply to dy, then the inte-

gral y being between the limits $y=y$ and $y=0$ (see fig. 36), the above expression is the same as $du=ydx$. Hence, by writing du in the above general form, the expressions for X and Y will be

$$X=\frac{\iint x\,dy\,dx}{\iint dy\,dx},\ Y=\frac{\iint y\,dy\,dx}{\iint dy\,dx},$$

in which the integrations may be performed in any order. If in the first of these we integrate for y, first, we shall have the form (1) above: and if we integrate the second for x first, we shall have the form (2), but if we integrate this for y first, then $Y=\frac{\frac{1}{2}\int y^2dx}{\int ydx}$. (3).

When the area is bounded by straight lines only, the case will be too simple to require the aid of the calculus, as in the first of the following problems.

PROBLEM V.—To determine the centre of gravity of a plane triangle ABC (fig. 37).

Bisect any side as AB by a line CD from the opposite angle, then the centre of gravity must be in this line; for it will bisect any line $a\,b$ whatever drawn parallel to AB, so that, taking any point in $d\,a$, there shall always be a corresponding point in $d\,b$, equidistant from CD, or from a plane through CD perpendicular to the plane of the triangle; hence the sum of the moments on one side this plane will be equal to the sum of the moments on the other side; hence the centre of gravity being necessarily in the plane of the figure, must be in the line CD. In like manner, if AC be bisected by the line BE, the centre will lie in this line, hence it is at the point G where it intersects the former.

We might immediately infer from this, that the three lines from the angles of a triangle bisecting the opposite sides necessarily meet in a point. To determine the distance of this point from C let us draw ED, which, as it bisects the two sides AB, AC, must be parallel to CB and equal to half of it; hence the triangles EGD, BCD are similar $\therefore \frac{BC}{ED}=\frac{CG}{GE}=2 \therefore CG=\frac{2}{3}CE$; to express this distance analytically, put a, b, c, for the sides respectively opposite to the angles A, B, C, and e for the line CE, then, (*Geometry*, p. 38-9,)

$$2e=\sqrt{\{2a^2+2b^2-c^2\}} \therefore CG=\tfrac{1}{3}\sqrt{\{2a^2+2b^2-c^2\}}$$
$$\text{in like manner } BG=\tfrac{1}{3}\sqrt{\{2a^2+2c^2-b^2\}}$$
$$AG=\tfrac{1}{3}\sqrt{\{2b^2+2c^2-a^2\}}$$

Adding together the squares of these expressions there results the equation $3\,(AG^2+BG^2+CG^2)=AB^2+BC^2+CA^2$; that is, in any plane triangle the sum of the squares of the sides is equal to three times the sum of the squares of the distances of the vertices from the centre of gravity of the triangle.

If three equal bodies be placed at the vertices of a triangle, their centre of gravity will coincide with the centre of gravity of the triangle; for the centre of gravity of the two equal bodies A, B will be at D, and the weight at this point will be 2 A; hence, joining D, C, the centre of gravity of 2 A and C = A will be a point G, such that $\frac{2\text{ A}}{\text{A}} = \frac{\text{CG}}{\text{GD}}$.

Again, the centre of gravity will still be the same point if the equal weights be placed at the middle points D, E, F; for the middle M of DE will be the centre of gravity of two of them, and the centre of these and the third will be a point G, such that MG= $\frac{1}{2}$GF and MG *is* equal to $\frac{1}{2}$GF, because, by similar triangles,

$$\text{EGM, BGF}, \ \frac{\text{BF}}{\text{EM}} = \frac{\text{GF}}{\text{GM}} = 2.$$

The centre of gravity of any plane polygon may be found by first finding the centres of gravity of its component triangles, and then the common centre of gravity of all these points loaded with their respective triangles.

Problem VI.—To determine the centre of gravity of a circular segment (fig. 38).

From the equation of the curve, taking the centre for origin, and putting OD = a, we have $y = \sqrt{\{r^2 - x^2\}}$

$$\therefore \text{X} = \frac{\int_a^r xy\,dy}{u} = \frac{\int_a^r x\sqrt{\{r^2 - x^2\}}dx}{u} = \frac{\frac{2}{3}(r^2 - a^2)^{\frac{3}{2}}}{u}.$$

If the segment is a semicircle $a = 0$, and $u = 1{\cdot}57079\, r^2$, hence, in this case, $\text{X} = \frac{\frac{2}{3}r^3}{u} = \frac{r}{2\,.\,35619} = {\cdot}42441\, r.$

Problem VII.—To determine the centre of gravity of the common parabola (fig. 39).

From the equation of the curve, $y = \sqrt{\{px\}}$

$$\therefore \text{X} = \frac{\int xy\,dx}{\int y dx} = \frac{\int x^{\frac{3}{2}}\,dx}{\int x^{\frac{1}{2}}\,dx} = \frac{3}{5}\,x = \frac{3}{5}\,\text{A}x$$

If we require the centre of gravity of the semi-parabola ABx, then Y, which for the whole parabola is 0, will have to be calculated by the second formula (3), that is, we shall have

$$\text{Y} = \frac{\int y^2\,dx}{2\int y\,dx} = p^{\frac{1}{2}}\,\frac{\int x\,dx}{2\int x^{\frac{1}{2}}\,dx} = \frac{3}{8}\sqrt{\{px\}} = \frac{3}{8}\,y = \frac{3}{8}\,\text{B}x.$$

Determination of the Centre of Gravity of a Surface of Revolution.

(44.) Let the axis of x be the axis of revolution, then it will pass through the centre of gravity of the body, and therefore the first of

the general equations (A) will be sufficient to determine it. Putting S for the surface, this equation becomes, (*see Int. Calc.* p. 138,)

$$X=\frac{\int x dS}{S}=\frac{\int xy\, ds}{\int y ds}=\frac{\int xy\sqrt{1+\frac{dy^2}{dx^2}}\,dx}{\int y\sqrt{1+\frac{dy^2}{dx^2}}\,dx}$$

PROBLEM VIII.—To determine the centre of gravity of a spheric surface (fig. 40).

By the circle, $x^2+y^2=r^2$

$$\therefore \frac{dy^2}{dx^2}=\frac{x^2}{y^2}\therefore yds=rdx\therefore \frac{\int xy\, ds}{\int yds}=\frac{\int xdx}{\int dx}.$$

Hence, integrating between $x=r$ and $x=OC=a$, we have

$$X=\frac{\frac{1}{2}(r^2-a^2)}{r-a}=\frac{1}{2}(r+a)=OG;$$

hence G is at the middle of CB.

PROBLEM IX.—To determine the centre of gravity of a conical surface (fig. 41).

Taking the centre of the cone as the origin, we have for the revolving line OB the equation

$$y=ax\therefore\frac{dy^2}{dx^2}=a^2\therefore yds=a\sqrt{\{1+a^2\}}\,dx$$

$$\therefore\frac{\int xy\, ds}{\int y\, ds}=\frac{\int x^2\, dx}{\int x\, dx}=\frac{2}{3}x=\frac{2}{3}OC.$$

Hence G is at the distance of one third the height from the base.

Determination of the Centre of Gravity of a Solid of Revolution.

(45.) Putting V for B in the general expression for X, and we have for any volume of revolution $X=\frac{\int x\, dV}{V}=\frac{\int xy^2\, dx}{\int y^2\, dx}$.

PROBLEM X.—To determine the centre of gravity of a segment of a spheroid (fig. 42).

Taking the origin at A, the vertex of the generating ellipse, we have for its equation

$$y^2=\frac{b^2}{a^2}(2ax-x^2)\therefore X=\frac{\int xy^2\, dx}{\int y^2\, dx}=\frac{\int(2ax-x^2)\,xdx}{\int(2ax-x^2)\,dx},$$

that is $$AG=\frac{\frac{2}{3}ax^3-\frac{1}{4}x^4}{ax^2-\frac{1}{3}x^3}=\frac{8ax-3x^2}{12a-4x}.$$

For a hemispheroid we have $x=a$ and $X=\frac{5}{8}a$.

As the foregoing expression for AG is independent of b, it remains the same when $b=a$, or when the body is a sphere, so that if a sphere be described on either axis of a spheroid, any segments cut off by a plane perpendicular to this axis will have the same centre of gravity.

(46.) The general formula employed in this article will, after a slight modification, serve to determine the centre of gravity of any volume generated by the motion of a varying surface along a fixed axis, perpendicular to its plane, and passing through its centre of gravity, as the axis of x, provided only that in every position this surface is the same function of x. For, calling the generating plane K, the differential dB of the body generated will be $K\,dx$, (*Int. Calc.* p. 144,) and hence the formula in last article will be

$$X=\frac{\int K\,x dx}{\int K\,d\,x}$$

We shall give two examples of the application of this form of the general equation.

Problem XI.—To determine the centre of gravity of a pyramid or cone.

Let AB be the axis, which call b, and the area of the base Hh, a; then, since the area of any two sections perpendicular to the axis are as the squares of their distances from A, we have

$$\frac{b^2}{x^2}=\frac{a}{K}\therefore K=\frac{ax^2}{b^2},$$

substituting this value of K in the formula above, there results

$$X=\frac{\int Kx\,dx}{\int K\,dx}=\frac{\int x^3\,dx}{\int x^2\,dx}=\frac{3x^4}{4x^3}=\frac{3}{4}x;$$

hence AG is equal to $\frac{3}{4}$ the altitude of the cone or pyramid.

If the solid is the frustum or trunk of a cone or pyramid of which the distances of the two ends from A are respectively b', b, then, in the expression for x, we must integrate between these values of x, that is

$$X=\frac{\int_{b'}^{b} Kx\,dx}{\int_{b'}^{b} K\,dx}=\frac{\int_{b'}^{b} x^3\,dx}{\int_{b'}^{b} x^2\,dx}=\frac{3}{4}\cdot\frac{b^4-b'^4}{b^3-b'^3}.$$

Problem XII.—To determine the centre of gravity of any segment of an ellipsoid.

Calling the principal semiaxes a, b, c, the equation of the surface is $a^2 b^2 z^2+a^2 c^2 y^2+b^2 c^2 x^2=a^2 b^2 c^2$, and the equation of a section at the distance x from, and parallel to, the plane of yz, is

$$b^2 z^2+c^2 y^2=\frac{b^2 c^2 (a^2-x^2)}{a^2},$$

from which we get these functions of x for the semiaxes of the generating ellipse, viz.

$$b'=\frac{b\sqrt{\{a^2-x^2\}}}{a},\ c'=\frac{c\sqrt{\{a^2-x^2\}}}{a},$$

and therefore the area of the generating surface in any position is, (*Int. Calc.* p. 124,) $\pi b'c'=\frac{\pi bc}{x^2}(a^2-x^2)=\text{K}$

$$\therefore \frac{\int \text{K}x\,dx}{\int \text{K}\,dx}=\frac{\int x(a^2-x^2)\,dx}{\int (a^2-x^2)\,dx}=\frac{\frac{1}{2}a^2x^2-\frac{1}{4}x^4}{a^2x-\frac{1}{3}x^3}=\frac{6a^2-3x^2}{12a^2-4x^2}x.$$

This expression being independent of b and c, shows us that ellipsoids having one common axis, have a common centre of gravity for all the segments cut off by a plane perpendicular to that axis; which is an extension of the property noticed at (45).

(47.) Having now given, in succession, all the most usual formulas for the determination of the centre of gravity, together with their practical illustration, we shall terminate by merely writing down the forms which the general equations (A) take when we wish to apply them to a surface, or a volume which is not of revolution, nor yet symmetrical with respect to an axis.

The general expression for the differential of a surface S, referred to three rectangular planes, is, (*Int. Calc.* p. 151,)

$$d\,\text{S}=\int\sqrt{1+\frac{dz^2}{dx^2}+\frac{dz^2}{dy^2}}\cdot dx\,dy;$$

where it is optional for us to consider the differential dS to be taken either with respect to x or with respect to y; hence putting this for dB in the general equations referred to, or rather writing the coefficients under the radical in the more abridged form p', q', we have these expressions for the co-ordinates of the centre of gravity of the surface,

$$\text{X}=\frac{\iint x\sqrt{1+p'^2+q'^2}.\,dx\,dy}{\iint\sqrt{1+p'^2+q'^2}.\,dx\,dy}$$

$$\text{Y}=\frac{\iint y\sqrt{1+p'^2+q'^2}.\,dx\,dy}{\iint\sqrt{1+p'^2+q'^2}.\,dx\,dy}$$

$$\text{Z}=\frac{\iint z\sqrt{1+p'^2+q'^2}.\,dx\,dy}{\iint\sqrt{1+p'^2+q'^2}.\,dx\,dy}.$$

Again, the general expression for the differential of a volume V is, (*Int. Calc.* p. 148,) $d\text{V}=\int z\,dy\,dx$; in which it is optional whether we consider the differentiation to be performed with respect to the independent variable x or y; but we may render this expression still more general, as well as more symmetrical by putting $\int dz$ for z, for then the differential $d\text{V}=\iint dz\,dy\,dx$ may be considered

as taken relatively to either of the three variables x, y, z, we please. Hence the expressions for the co-ordinates of the centre of gravity of the volume are

$$X=\frac{\iiint x\,dz\,dy\,dx}{\iiint dz\,dy\,dx}\text{ or }\frac{\iint xz\,dx\,dy}{\iint z\,dx\,dy}$$

$$Y=\frac{\iiint y\,dz\,dy\,dx}{\iiint dz\,dy\,dx}\text{ or }\frac{\iint yz\,dy\,dx}{\iint z\,dy\,dx}$$

$$Z=\frac{\iiint z\,dz\,dy\,dx}{\iiint dz\,dy\,dx}\text{ or }\frac{\frac{1}{2}\iint z^2\,dy\,dx}{\iint z\,dy\,dx}.$$

On Guldin's Theorem, or the Centrobaryc Method.

(48.) The expressions for Y, at articles (42) and (43), furnish a very remarkable theorem for the determination of the surfaces and volumes of bodies of revolution. The expressions referred to immediately give the equations

$$2\pi Ys=2\pi\int y\,ds \text{ and } 2\pi Y\int y\,dx=\pi\int y^2\,dx;$$

the former relating to the curve line, the latter to the surface. Now $2\pi Y$ is the circumference of which Y is the radius; it, therefore, expresses the circumference which would be described by the centre of gravity of the line s if it were to revolve round the axis of x: but $2\pi\int y\,ds$ expresses the area of the surface which would be engendered by this revolution: hence, 1. *The surface generated by the revolution of a curve round an axis is equal to the length of that curve multiplied by the circumference described by the centre of gravity.*

Again, $2\pi Y$, in the second of the above equations, being equal to the circumference which would be described by the centre of gravity of the surface s, if it were to revolve round the axis of x, and $\pi\int y^2dx$ being the expression for the very volume which would thus be generated, it follows that, 2. *The volume generated by the revolution of a plane surface round an axis is equal to the area of that surface multiplied by the circumference described by its centre of gravity.*

These two propositions comprise the theorem of *Guldin*, and their application to the determination of the surfaces and volumes of bodies constitutes the *Centrobaryc method.* By this method we see that when we know the length of the generating line, or the area of the generating surface, as also the distance of its centre of gravity from the axis of revolution, the value of the surface, or solid generated either by a whole or a partial revolution, may be at once found. Also any two of these three things being given, viz. the generatrix, the distance of its centre of gravity from the axis, and the magnitude generated being given, the third may be found.

As an example of this method, suppose we wanted to know the volume of a paraboloid of revolution: Let a be the axis or height of the generating semi-parabola, and b its base; then the distance of its centre of gravity from the axis $\frac{3}{8}b$, so that the circumference generated by this centre is $\frac{3}{4}b\pi$. Again, the area of the generating surface is, (*Int. Calc.* art. 67,) $\frac{2}{3}ab$, consequently, multiplying these two quantities together, have, for the volume sought,

$$V=\frac{3}{4}b\,\pi\times\frac{2}{3}ab=\frac{1}{2}ab^2\,\pi;$$

this volume is, therefore, $\frac{1}{2}$ that of its circumscribing cylinder.

Suppose now that we wished to determine by this method the centre of gravity of a semicircle of radius r. We know that the area of this semicircle is $\frac{1}{2}r^2\pi$, and that the volume of the sphere, generated by it is $\frac{4}{3}r^3\pi$; hence, as this expression must be equal to the former multiplied by the circumference described by the centre of gravity sought, we have for this circumference, the value

$$\frac{\frac{4}{3}r^3\pi}{\frac{1}{2}r^2\pi}=\frac{8}{3}r;\ \therefore\ \frac{8}{3}r\div 2\,\pi=\cdot 42441\,r=X;$$

and this is a more simple way of determining the centre than that employed in problem VI.

We shall conclude the present chapter with a few miscellaneous examples for the exercise of the student.

1. For the centre of gravity of a parabola of the nth order, whose equation is $a^{n-1}y=x^n$; the expression is $X=\frac{2n+1}{2n+2}x$.

2. For the distance of the centre of gravity of a semi-ellipse whose axes are $2a$, $2b$, from the base or minor axis, the expression is

$$X=\frac{4\,a}{3\,\pi}.$$

3. For a paraboloid of revolution, whose altitude is a, $X=\frac{2}{3}a$.

4. For a segment of hyperboloid, whose altitude is a,

$$X=\frac{8a+3x}{12a+4x}x.$$

5. The convex surface of a conic frustum, or trunk, is found by Guldin's theorem to be equal to half the sum of the circumferences of the ends multiplied by the slant height.

SCHOLIUM.

It has been shown in the outset of this chapter, that for a body to be supported, it is absolutely necessary that its centre of gravity lie in a vertical line, passing through the base on which the body stands;

or if the body stand on props or legs, this vertical line must pass through the area, which a string stretched round these legs would enclose. The space thus enclosed by the feet of the human body is, obviously, but small, and when we consider the very various positions in standing, stooping, walking, &c. which we can easily and safely assume, and the great rapidity with which we can pass from any one of these positions into another, we cannot fail to be impressed with the wisdom and bounty of the great Creator, who, by such admirable disposition of the limbs and joints of the body has rendered this small space sufficient for its support in all these various attitudes. A person in danger of falling experiences an irresistible impulse to overtake, as it were, the point where the line of direction seeks to meet the horizontal plane, and hence the long strides which he is impelled to make when violently pushed in the back, the feet endeavouring to overstep the point alluded to; so when standing and inclining the body forward, till the line of direction is about to fall beyond our toes, we cannot help putting forward our foot to overstep it. Children who have not the same command over their limbs as grown persons, and who have, moreover, a less sense of danger, do not always use the best means to recover their stability when about to fall, but then in the more hazardous circumstances they usually take care to secure for themselves a more spacious base than grown persons; thus in walking up or down stairs, we commonly see children employ both their hands and feet, thus securing for themselves a very considerable base, out of which the line of direction is not easily forced.

If a body rest on a single point, it is necessary that the vertical line through it should pass also through the centre of gravity, but "in certain cases a body resting upon a single point may yet have a disposition to recover from any derangement, and to resume its vertical position. Thus if the base be a plane, and the bottom of the body rounded, but such that the centre of gravity lies below the centre of curvature, the mass may rock backwards and forwards, but will soon regain its erect site. Let O (fig. 43,) be the centre of the incurvation at the end of the body, and G or g its centre of gravity lying in the axis AO. Conceive the body to be rolled on its horizontal plane from A to A′, the point which touched A will merge into a, and the axis will come into the position aO′. Now if the centre of gravity G stood above O, it would evidently in the position G′ lean beyond the vertical A′O′, and the body would fall over: but if the centre of gravity were at g below O, it would still in changing to g' lie within the vertical A′O′, and, consequently, the body would roll back to its first position." *Leslie's Natural Philosophy*, p. 55.

It may be remarked, that when, as in the case just adduced, the body resists the tendency to overturn it, and returns to its first po-

sition of equilibrium, this equilibrium is called *stable*; but when it yields to every tendency to overturn it, and falls, the equilibrium from which it has been disturbed is called *unstable*. The equilibrium of an ellipse resting on the extremity of its minor diameter is stable, but the equilibrium when it rests on the extremity of the major diameter is unstable.

CHAPTER III.

ON THE EQUILIBRIUM OF A SOLID BODY ACTED UPON BY FORCES APPLIED TO DIFFERENT POINTS AND IN DIFFERENT DIRECTIONS.

(49.) WE come now to consider a body or system of material points connected together in an invariable manner, when in a state of equilibrium from the action of any system of forces whatever, and to determine what the conditions are which must necessarily characterize such a system. The reasonings, therefore, of this chapter will be so general that the results to which they lead will comprehend in them all the particulars hitherto deduced respecting the equilibrium of forces acting under certain restrictions, whether through the intervention of a solid body, or upon a single point. The more important of these particular deductions are, however, essential to the establishment of the general theory, so that we are not to expect that the discussion of the general proposition, on which we are about to enter, will be independent of its particular cases already considered, but that it will on the contrary be in a great measure founded upon them. We shall find it convenient to divide this proposition into two parts, taking first the case where the component forces all act in one plane, and afterwards considering them to act without restriction.

I. *When the forces are all situated in one plane.*

(50.) Let the plane of the forces P, P_1, P_2, &c. be taken for the plane of xy, and let their points of application be (x, y), (x_1, y_1), (x_2, y_2), &c., also let the inclinations of the forces to the axis of x be as usual, α, α_1, α_2, &c., and to the axis of y, β, β_1, β_2, &c.; these latter angles being the complements of the former. Let now each force be decomposed into two parallel to the axis, and we shall thus have instead of

P the components P cos. α, P cos. β
P_1 . . . P cos. α_1, P cos. β_1
P_2 . . . P cos. α_2, P cos. β_2
&c. &c.

so that the points are now acted upon by two systems of parallel forces, and, supposing that each system has a single resultant, it will be parallel to the components, and it follows that the original system may be replaced by two forces X and Y, of which the intensities are

$$\left.\begin{array}{l} X=P \cos. \alpha+P_1 \cos. \alpha_1+P_2 \cos. \alpha_2+\&c. \\ Y=P \cos. \beta+P_1 \cos. \beta_1+P_2 \cos. \beta_2+\&c. \end{array}\right\} \quad \ldots \ldots (1);$$

thus when we know the right-hand members of these equations, as we are here supposed to do, we know also the intensities and directions of the two forces X, Y, but not as yet their points of application.

To determine these let y' be the distance of the force X, from the axis of x, to which it is parallel, and x' be the distance of Y from the axis of y, then we know (38) that

$$\left.\begin{array}{l} X y'=y\, P \cos. \alpha+y_1\, P_1 \cos. \alpha_1+y_2\, P_2 \cos. \alpha_2+\&c. \\ Y x'=x\, P \cos. \beta+x_1\, P_1 \cos. \beta_1+x_2\, P_2 \cos. \beta_2+\&c. \end{array}\right\} \quad \ldots \ldots (2);$$

from which, as the values of x, x_1...., y, y_1.... are supposed to be known, x' and y' become known, and two lines drawn parallel to and at these distances from the axes will coincide with the directions of the forces X, Y, and as a force may be applied at any point of its direction, we may consider the intersection of these two lines to be the common point of application of the two forces X, Y, so that the original system is thus reduced to two determinate forces, acting on a determinate point in known directions; and, lastly, these are reducible to a single determinate force R, by the equations

$$R=\sqrt{\{X^2+Y^2,\}} \cos. a=\frac{X}{R}, \cos. b=\frac{Y}{R} \quad \ldots \ldots (3);$$

a and b being the inclinations of R to the axes of x and y.

It is thus proved, that when a system of forces, acting with given intensities, and in given directions, upon a given system of points invariably connected, has a single resultant, its intensity, direction, and point of application may all be determined by means of the given quantities.

(51.) It should be remarked here, that the equations (2) which enable us to determine the point (x', y') of application of the resultant, after we have found X and Y, furnish us with more information than we absolutely require, for it would be quite sufficient for us to know the situation of *any* point through which the resultant passes, for knowing this the position and intensity of it would be given by equations (3), and the effect of it will, we know, be the same to whatever point it be applied. Again the position thus determined ought, obviously, to be independent of the co-ordinates x, y; x_1, y_1; &c. because it would remain the same to whatever points the forces

P, P_1, &c. are applied, provided we take them in their directions. What is here said amounts to this, viz. that we ought to be able to determine the perpendicular distance of the resultant from a given fixed point, when we know the perpendicular distances of the components from that point, and this determination we shall be able to effect by means of the equations (2).

The given fixed point from which the distances of the directions of the several forces are to be estimated, we shall, for simplicity, consider to be the origin and the product of each force by the perpendicular on its direction, we shall call the *moment of that force* with respect to the proposed point. This being premised, let us substitute for X and Y, in equations (2), their values (3), viz. $R \cos. a$ and $R \cos. b$, and then subtract the one equation from the other, and we shall thus have

$$R\,(y' \cos. a - x' \cos. b) = P\,(y \cos. \alpha - x \cos. \beta) + P_1\,(y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + \&c.$$

Let us inquire into the meaning of the first side of this equation. Suppose m (fig. 44) to be the point of application of the resultant $R = m\,n$, then $mB = x'$, $m\,A = y'$, and let perpendiculars be drawn from O, A, B, on the resultant, then it is plain that $y' \cos. a = A\,p$, $x' \cos. b = B\,q$; also, if Os be drawn parallel to $m\,p$ the triangles, mqB, OsA, will be in all respects equal, and, therefore, $As = Bq$, therefore $Ap - Bq = Or$, which call r, consequently

$$R\,(y' \cos. a - x' \cos. b) = R\,r = \textit{moment of } R \ldots . (4).$$

In like manner, for any other force P_n, we must have

$$P_n\,(y_n \cos. \alpha_n - x_n \cos. \beta_n) = P_n\,p_n = \textit{moment of } P_n \ldots . (5);$$

p_n representing the perpendicular from O on the direction of P_n; hence the foregoing equation is the same as

$$R\,r = P\,p + P_1\,p_1 + P_2\,p_2 + P_3\,p_3 + \&c. \ldots . (6);$$

so that *the moment of the resultant is equal to the sum of the moments of the components.*

In the figure to which our reasoning has been applied we have so taken the direction of the force R that the angles a and b may have positive cosines: but the inference (4) will always be the same whatever be the directions of R, thus in fig. 45, where cos. b is negative, the same reasoning as that employed above shows us that $Or = Bq + Ap$, and thus the same conclusion (4) is obtained.

(52.) The foregoing general theorem is analagous, in terms, to that given at (38), for the moments of a system of parallel forces with respect to a plane; but the student must carefully observe that there is no analogy between the theorems themselves; for the moment of a force, with respect to a point, is altogether distinct from its moment with respect to a plane; this latter moment depends on the point of application of the force, and is independent of its direc-

tion; whereas the moment, with respect to a point, on the contrary, depends on the direction of the force, and is independent of its point of application.

The second side of the equation (6) is supposed to be entirely known, and this is only supposing that the intensities P, P_1, &c. are known, and the several directions in which they act, or which is the same thing, the equations of these directions. Thus, suppose the equation of P_n's direction is known to be

$$y=\frac{\cos.\ \beta_n}{\cos.\ \alpha_n}x+b \ \therefore\ b\cos.\ \alpha_n=y\cos.\ \alpha_n-x\cos.\ \beta_n,$$

x and y being co-ordinates of any point whatever in the given direction; if we take any particular point (x_n, y_n) we have

$$b\cos.\ \alpha_n=y_n\cos.\ \alpha_n-x_n\cos.\ \beta_n;$$

therefore, by equation (5), last article $p_n=b\cos.\ \alpha_n$; and b and $\cos.\ \alpha_n$ are, by hypothesis, known both as to sign and quantity; hence p_n is known both as to sign and quantity. Hence the second member of the equation (6) is entirely known when the intensities and direetions of the forces are known; also R is known, from the same data, by the equations (1) and the first of (3) in article (50), consequently, r, and, therefore, the position of the resultant R is known. By using the notation employed at (39) the expression for r will be

$$r=\frac{\Sigma(\mathrm{P}p)}{\mathrm{R}};$$

and the equations necessary for the determination of the resultant, both in intensity and direction, will be those marked (1) and (6) in last article; that is the three following

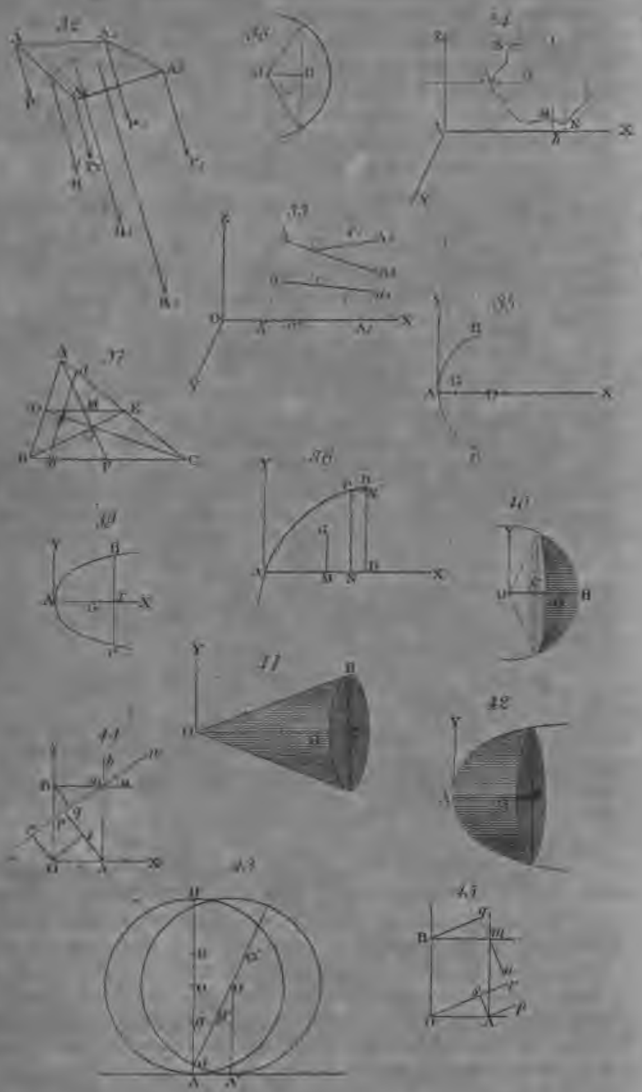
$$\mathrm{X}=\Sigma\,(\mathrm{P}\cos.\ \alpha),\ \mathrm{Y}=\Sigma\,(\mathrm{P}\cos.\ \beta),\ \mathrm{R}r=\Sigma\,(\mathrm{P}p);$$

the first two giving $\mathrm{R}=\surd\{\mathrm{X}^2+\mathrm{Y}^2\}$.

(53.) When the system is in equilibrium then we have $\mathrm{X}=0$, $\mathrm{Y}=0$, and, consequently, $\mathrm{R}=0$, so that the equations of equilibrium are $\Sigma\,(\mathrm{P}\cos.\ \alpha)=0$, $\Sigma\,(\mathrm{P}\cos.\ \beta)=0$, $\Sigma\,(\mathrm{P}p)=0$; or, in their more expanded form,

$$\left.\begin{array}{l}\mathrm{P}\cos.\ \alpha+\mathrm{P}_1\cos.\ \alpha_1+\mathrm{P}_2\cos.\ \alpha_2+\&c.=0\\ \mathrm{P}\cos.\ \beta+\mathrm{P}_1\cos.\ \beta_1+\mathrm{P}_2\cos.\ \beta_2+\&c.=0\\ \mathrm{P}p\quad\ +\mathrm{P}_1p_1\quad\ +\mathrm{P}_2p_2\quad\ +\&c.=0\end{array}\right\}\ .\ .\ (1).$$

We have already shown how the signs and values of the several products in the last of these equations are to be determined from the data of the problem; but, in the case of equilibrium, it sometimes is convenient to determine the signs from other considerations less analytical. Let us suppose the origin of the axes A (fig. 46), which we have taken for the *centre of moments*, to be connected with the system by means of the rigid perpendiculars p, p_1, p_2, &c. on the directions of the several forces; then the whole system be-



ing at rest the point A will be at rest, and may, therefore, be considered as fixed. Now the tendency of the forces P, P_1, P_2, as marked in the figure, is to make the extremities of p, p_1, p_2, on which they act, to turn round A in the same direction, and, consequently, the tendency of the whole system with which these points are invariably connected is to turn round A; to prevent motion, therefore, the remaining forces P_3, P_4, &c. must impress on the system an equal tendency to turn round A in the opposite direction; we ought, therefore, to attach to these opposing tendencies contrary signs, that is, if we consider the forces P, P_1, P_2, &c. conspiring in one direction to be positive, we ought to consider the forces P_3, P_4, &c. conspiring in the opposite direction to be negative. In the former method of determining the sign of any moment $P_n p_n$, we considered P_n to have always the same sign, or to be positive, so that the moment always took the sign of p_n. If, on the contrary, we attach the sign to P_n, it ought, obviously, to be in conformity to the principle just stated, and then we must consider p_n to be always positive

(54.) Of the preceding equations of equilibrium we may now remark that the two first establish the condition that there must exist some *fixed* point, A, in connexion with the system, round which, however, the system *may* revolve; the third equation, in conjunction with these, must then establish the remaining condition of equilibrium, viz. that there can be no motion of rotation. If the body or system, on which the forces act, be not entirely free, but be only at liberty to move in any direction about a fixed point, then the last of the above equations will be sufficient to establish the equilibrium, the fixed point being taken for the centre of moments.

If of the equations (1) only the last has place, then all we can infer is that the system has no tendency to move round the point taken for the centre of moments, and since we here suppose that the value of R, viz. $R=\sqrt{\{X^2+Y^2,\}}$ is something, but that Rr or $\Sigma\,(Pp)$ is 0, we must conclude that $r=0$, so that the point taken for the centre of moments must be on the resultant, in order that the equation $\Sigma\,(Pp)=0$ may have place, and for every point on the resultant it obviously will have place; so that if any point on the resultant were to be fixed, this point would sustain the whole pressure of the system.

II. When the forces are situated in different planes.

(55.) Let us now consider the forces P, P_1, P_2, &c. as acting, in any manner whatever, upon a solid body, or upon a system of points, invariably connected. Let the inclinations of P to the axes of x, y, and z, be as usual α, β, γ; the inclinations of P_1, α_1, β_1, γ_1; and so on. Moreover let the co-ordinates of any point in P's di-

rection be x, y, and z; those of any point in P_1's direction, x_1, y_1, z_1, &c; and let all these directions be produced till they pierce one of the co-ordinate planes, say the plane of xy, excluding for the present those forces which may be parallel to this plane. Let us call the co-ordinates of the point where P pierces the plane of xy, x', and y', then, for the equations of this line, we shall have, (*Anal. Geom.*, p. 225-6,)

$$\left.\begin{array}{l} x-x'=a\,z \\ y-y'=b\,z \end{array}\right\} \quad \ldots\ldots (1);$$

where a and b denote the tangents of the angles which the projections of the proposed line on the vertical planes make with the axis of z. Now between these constants a, b, and the given inclinations of the line itself to the axes, there exists the relations, (*Anal. Geom.* p. 228,)

$$\cos.\,\alpha=a\cos.\,\gamma\,,\ \cos.\,\beta=b\cos.\,\gamma$$

$$\therefore a=\frac{\cos.\,\alpha}{\cos.\,\gamma}\,,\ b=\frac{\cos.\,\beta}{\cos.\,\gamma};$$

hence substituting these values in the equations (1), and putting $z=0$, we have, for the co-ordinates, x', y', of the point where P pierces the plane of xy, the values

$$x'=\frac{x\cos.\,\gamma-z\cos.\,\alpha}{\cos.\,\gamma}=\frac{x\,P\cos.\,\gamma-z\,P\cos.\,\alpha}{P\cos.\,\gamma}\ \ldots\ldots (2)$$

$$y'=\frac{y\cos.\,\gamma-z\cos.\,\beta}{\cos.\,\gamma}=\frac{y\,P\cos.\,\gamma-z\,P\cos.\,\beta}{P\cos.\,\gamma}\ \ldots\ldots (3);$$

at this point (x', y'), since it is in P's direction, we may consider P to be applied; let us then do so and decompose P thus applied, according to the three axes, and we shall then have, instead of the original force P, these three, viz. P cos. α, P cos. β, P cos. γ; the first two acting in the plane of $x\,y$, and the third acting parallel to the axis of z, and all upon the point (x', y') where P's original direction pierces the horizontal plane. Hence the moments of the vertical force P cos. γ with respect to the vertical planes, are, by taking account of equations (2) and (3) just given,

$$x'\,P\cos.\,\gamma=P\cos.\,\gamma-z\,P\cos.\,\alpha$$

$$y'\,P\cos.\,\gamma=y\,P\cos.\,\gamma-z\,P\cos.\,\beta.$$

Now whatever we have said respecting the force P and its components, equally applies to any other force P_n, that is, it is equivalent to the three component forces $P_n\cos.\,\alpha_n$, $P_n\cos.\,\beta_n$, $P_n\cos.\,\gamma_n$;

acting on the point where P_n's direction pierces the horizontal plane: the first two being in that plane and parallel to the axis of x and y, and the third force being vertical, or parallel to the axis of z; moreover the moments of this last force, in reference to the vertical planes, must be

$$x_n\,P_n\cos.\,\gamma_n-z_n\,P_n\cos.\,\alpha_n \text{ and } y_n\,P_n\cos.\,\gamma_n-z_n\,P_n\cos.\,\beta_n;$$

Consequently, the original system of forces may be reduced to three distinct parallel systems; one system being vertical, and the other two both in the horizontal plane, one of these parallel to the axis of x, the other to the axis of y, and the combinations of these systems must equilibrate if the original forces equilibrate; but, in order to this, it is necessary that the vertical forces of themselves equilibrate, or have no resultant, because no vertical resultant could be counteracted by the horizontal forces, hence the vertical and horizontal forces equilibrate separately. The vertical forces, therefore, must fulfil the conditions at page 57, that is, they must satisfy the equations $\Sigma(\mathrm{P}\cos.\gamma)=0$, $\Sigma(x\,\mathrm{P}\cos.\gamma - z\,\mathrm{P}\cos.\alpha)=0$, $\Sigma(y\,\mathrm{P}\cos\gamma - z\,\mathrm{P}\cos.\beta)=0 \ldots\ldots (4)$. The horizontal forces being also in equilibrium they must satisfy the equations (1) at art. (53), viz.

$$\Sigma(\mathrm{P}\cos.\alpha)=0,\ \Sigma(\mathrm{P}\cos.\beta)=0,\ \Sigma(\mathrm{P}p)=0 \ldots\ldots (5),$$

where it must be observed that $\mathrm{P}p=y'\,\mathrm{P}\cos.\alpha - x'\,\mathrm{P}\cos.\beta$; and, by the equations (2) and (3) above, the second side of this equation is the same as $y\,\mathrm{P}\cos.\alpha - x\,\mathrm{P}\cos.\beta$. Hence, writing the equations (4) and (5) in their expanded forms, we have these six equations of equilibrium, viz.

$$\left.\begin{array}{l}\mathrm{P}\cos.\alpha+\mathrm{P}_1\cos.\alpha_1+\mathrm{P}_2\cos.\alpha_2+\&c.=0\\ \mathrm{P}\cos.\beta+\mathrm{P}_1\cos.\beta_1+\mathrm{P}_2\cos.\beta_2+\&c.=0\\ \mathrm{P}\cos.\gamma+\mathrm{P}_1\cos.\gamma_1+\mathrm{P}_2\cos.\gamma_2+\&c.=0\end{array}\right\}\ldots\ldots(6),$$

$$\left.\begin{array}{l}(y\,\mathrm{P}\cos.\alpha-x\,\mathrm{P}\cos.\beta)+(y_1\,\mathrm{P}_1\cos.\alpha_1-x_1\,\mathrm{P}_1\cos.\beta_1)+(y_2\,\mathrm{P}_2\cos.\alpha_2-x_2\,\mathrm{P}_2\cos.\beta_2)=0\\ (x\,\mathrm{P}\cos.\gamma-z\,\mathrm{P}\cos.\alpha)+(x_1\,\mathrm{P}_1\cos.\gamma_1-z_1\,\mathrm{P}_1\cos.\alpha_1)+(x_2\,\mathrm{P}_2\cos.\gamma_2-z_2\,\mathrm{P}_2\cos.\alpha_2)=0\\ (y\,\mathrm{P}\cos.\gamma-z\,\mathrm{P}\cos.\beta)+(y_1\,\mathrm{P}_1\cos.\gamma_1-z_1\,\mathrm{P}_1\cos.\beta_1)+(y_2\,\mathrm{P}_2\cos.\gamma_2-z_2\,\mathrm{P}_2\cos.\beta_2)=0\end{array}\right\}(7).$$

(56.) Let us inquire a little into the meaning of these two groups of equations.

The first of the group (6) shows us that in the case of equilibrium, the resultant of that set of components which is parallel to the axis of x, must be 0; and, in order to this, it is necessary that the resultant of those forces which pull one way must be equal to the resultant of those which pull the contrary way; these equal and contrary resultants may either act upon the same point, and thus mutually oppose each other, or they may act at different points, but both in the same plane as the line joining those points. If the forces we are now considering were all that acted on the system, then, in the first of these cases, the whole system would be kept in equilibrium, and the body would be at equal liberty to obey any new influences as if these forces had not been applied. But in the second case, where the contrary resultants of the systems of parallel forces do

not concur, the body would not equilibrate, but would have a tendency to revolve round a *fixed* point in the middle of the line, joining the points of application of the two resultants. As, however, the equation under consideration, the first of (6), thus far fixes a point in the system, it prevents the body from tending to move in the direction of the axis of x. Applying the same reasoning to each of the other two equations of the group (6), we infer that they are sufficient to prevent in the body all tendency to motion in the direction of the axes of y and z, so that the entire system cannot tend to change its place, that is, there can be no *motion of translation.*

There may, however, exist, in conjunction with these conditions, a tendency to *rotation* in three distinct directions, and these three tendencies must be counteracted, before the equilibrium of the body can be established. It must be these tendencies then that are counteracted by the remaining three conditions (7); but let us examine into the meaning of these equations, and, for this purpose, it will be sufficient to consider the first term y P cos. α — x P cos. β. The forces P cos. α, P cos. β, act perpendicularly to each other on the point to which P is originally applied, and in a plane perpendicular to the axis of z; y is the distance of that point, in the axis of z, which is in this plane, from the direction of the force P cos. α, and x is the distance of the same point from the direction of the force P cos. β; so that if these distances were rigid lines, these two forces would tend to turn the system about this point, that is about the axis of z in contrary directions; hence the expressions y P cos. α and x P cos. β are the moments of the forces P cos. α and P cos. β, with respect to the axis of z, or, which is the same thing, with respect to the point where their plane cuts the axis of z. Hence the group of equations (7) intimate that *in the case of equilibrium, the sum of the moments of the component forces, with respect to the three axes, are severally* 0; understanding by the moment of a force, with respect to an axis, the product of that force by the distance of the axis from its direction, the force itself always acting in a plane perpendicular to the axis. We have thus a right to infer, seeing that equations (6) and (7) fully establish the equilibrium, that when there is no motion of translation, and when moreover the moments of rotation about three rectangular axes are each 0, the moments of rotation about any other axes whatever must be 0.

It remains to consider the effect of those among the original forces which may be parallel to the plane of xy, since the decomposition we have hitherto employed supposes the forces all to meet this plane. Suppose P to be parallel to the plane of xy; at its point of application apply also two equal but opposite vertical forces Q, — Q, then one of these being directed towards the plane of xy, the component of this and P will meet that plane.

Call this resultant P′, then, instead of the force P, we shall have the two forces P′ and Q acting at the same point, and to which the preceding theory is applicable. Now the horizontal components of P′ are the same as the horizontal components of P, seeing that P′ is composed of P and a vertical force; and therefore the two first of equations (6) and the first of (7) remain unaltered, whether we substitute the two forces P′ and Q for P or not. For the third equation of (6), these two forces furnish the terms $P' \cos. \gamma + Q = -Q + Q = 0$, the same as the original force P for which $\cos. \gamma$ is 0. For the second of equations (7), these same forces furnish the terms $(x P' \cos. \gamma - z P' \cos. \alpha) = (-x Q - z P \cos. \alpha)$ and $(x Q - 0)$, the aggregate of which is the term $-z P \cos. \alpha$, the same that would be given by the original force P, for which $\cos. \gamma$ is 0; and precisely in the same way is it to be shown, that there will be no difference in the last equation whether we substitute for P the forces P′, Q, or take the force P itself. Hence the foregoing equations establish the equilibrium, however the original forces may act on the system.

(57.) When the body to which the forces are applied is not free, but is only at liberty to move in any direction round a fixed point, then, taking this point for the origin, the equilibrium will be established if the equations (7) have place, for these equations forbid all tendency to move in the only way in which the body is at liberty to move.

In such a case the pressure on the point must be equal in intensity to the resultant of all the applied forces, or to the resultant of the same forces if they were all to be transferred parallel to their directions to the fixed point itself, since, as is plain from equations (6), it will require the same pressure or opposing force to preserve the equilibrium in one case as in the other. Calling the pressures on the point, estimated in the several directions of the axes of x, y, and z, X, Y, and Z, we have, from equations (6),

$$X = -\Sigma (P \cos. \alpha),\ Y = -\Sigma (P \cos. \beta),\ Z = -\Sigma (P \cos. \gamma).$$

When the body can move only round a fixed axis, then, taking this for the axis of z, all tendency to this motion will be prevented if the single equation

$$(y P \cos. \alpha - x P \cos. \beta) + (y_1 P_1 \cos. \alpha_1 - x_1 P_1 \cos. \beta_1) + (y_2 P_2 \cos. \alpha_2 - x_2 P_2 \cos. \beta_2) + \&c. = 0, \text{ exist.}$$

We may consider the fixed axis to be secured at its extremities by two fixed points or pivots, and then the general equations of equilibrium will furnish us with expressions for the pressures sustained by these pivots. Call the pressures upon one of the points, $(0, 0, Z',)$ in the directions of the axes, X', Y', Z'; and the pressures upon the other point, $(0, 0, Z'',)$ X'', Y'', Z''; then, these being the forces which, with those applied to the body, secure

the equilibrium of the system, we must have, between them and the applied forces, the conditions

$X'+X''=-\Sigma(P\cos.\alpha), Y'+Y''=-\Sigma(P\cos.\beta)$, and $Z'+Z''=-\Sigma(P\cos.\gamma)$. (1), in order that the equations (6) may be satisfied; and likewise the conditions

$$\left.\begin{array}{l} z'X'+z''X''=-\Sigma(x P\cos.\gamma - z P\cos.\alpha) \\ z'Y''+z''Y''=-\Sigma(y P\cos.\gamma - z P\cos.\beta) \end{array}\right\}\ldots.(2),$$

in order that the equations (7) may be satisfied.

The third of the equations (1) shows that the axis is pressed in the direction of its length by the force $\Sigma(P\cos.\gamma)$; which is, therefore, divided between the two pivots, although there is nothing to teach us in what proportion. The remaining four equations of condition enable us to determine, when the intensities and directions, as well as the points of application of the applied forces, are given, what efforts they exercise on each pivot to carry away the axis.

(58.) Before concluding the present chapter, we shall briefly advert to the remarkable analogy which exists between the theory of moments and the projections of plane figures in geometry; and first we may remark, that the several terms which constitute the equations (7), and each of which we have called the moments of two component forces with respect to the axis perpendicular to their plane, may each be considered as the moment of the projection of the force itself on the plane perpendicular to the same axis, with respect to the origin A. Thus let us take any force P_n, and represent it by a part of its direction estimated from the point of application; then the projection of the line P_n on the plane of xy may be considered as the projection on that plane of the proposed force. It is obvious that the components of this projection parallel to the axes of x and y must be the very same as the components of P_n parallel to these axes, that is, this projection is the resultant of the two forces $P_n\cos.\alpha_n$ and $P_n\cos.\beta_n$; but when two forces act in a plane, the moment of the resultant in reference to a fixed point is equal to the sum of the moments of the components, or to their difference if they tend to turn the system in contrary directions about this point, and the moment of our two forces is $y_n P_n\cos.\alpha_n$, and $x_n P_n\cos.\beta_n$. Hence, calling the resultant of these forces, or the projection of P_n, P'_n; and p'_n the perpendicular upon it from the centre of moments, A, we have

$$p'_n P'_n = y_n P_n\cos.\alpha_n - x_n P_n\cos.\beta_n,$$

and similar expressions obviously have place for the moments of the projections on the other two planes.

As the projection of a line on a plane is the product of the line by the cosine of its inclination to the plane, or by the sine of its inclination to a perpendicular to the plane, the projection P'_n, of P_n on the plane of xy, will be expressed by the product $P_n\sin.\gamma_n$;

and, in like manner, the projections P''_n and P'''_n, of the same line on the planes xz and yz will be $P''_n = P_n \sin. \beta_n$, $P'''_n = P_n \sin. \alpha_n$; hence, calling the perpendiculars on these projections from the origin, or centre of moments, p''_n and p'''_n, we may write the equations (7) in this more concise form, viz.

$$\left.\begin{array}{l} p' \ P\sin.\gamma + p'_1 \ P_1 \sin.\gamma_1 + p'_2 \ P_2 \sin.\gamma_2 + \&c. = 0 \\ p'' \ P\sin.\beta + p''_1 \ P_1 \sin.\beta_1 + p''_2 \ P_2 \sin.\beta_2 + \&c. = 0 \\ p''' P\sin.\alpha + p'''_1 P_1 \sin.\alpha_1 + p'''_2 P_2 \sin.\alpha_2 + \&c. = 0 \end{array}\right\} \dots (7).$$

If p_n be the perpendicular from the origin, or centre of moments, upon the line P_n in space, then $p_n \ P_n$ will be double the area of the triangle whose base is P_n and vertex the origin; also $p'_n \ P'_n$ will be double the area of its projection; we may say, therefore, that the moment of the projection of any force, is equal to the projection of its moment.

As the moment $p' \ P'$ of the projection of any force P on the plane of xy, or the moment $p'' \ P''$, of the projection on the plane of xz, or the moment $p''' \ P'''$, of the projection on the plane of yz, is in each case equal to double the projection of the triangle whose base is P and vertex the origin, or centre of moments, it follows, from the theory of projections, (*Anal. Geom.* p. 256,) that if $p'''' \ P''''$ represent the moment of the projection of the same force on any fourth plane, passing through the centre of moments, we must have $\Sigma \ (p'''' \ P'''') = \cos. \delta \ \Sigma \ (p' P') + \cos. \delta' \ \Sigma \ (p'' P'') + \cos. \delta'' \Sigma \ (p''' \ P''')$, where δ, δ', δ'', are the inclinations of the new plane to the planes of $p' \ P'$, of $p'' \ P''$, and of $p''' \ P'''$ respectively.

It also follows from the same theory, (*Anal. Geom.* p. 256,) that if any number of forces be projected on three rectangular planes, and the moments of the projections on each plane, regarding the origin as the centre, be collected into one sum, the squares of the three sums thus furnished will be constant for every system of rectangular planes having the same origin.

For the determination of the *principal plane*, or that in which the moments of projection of any given forces amount to the greatest sum, see the *Analytical Geometry*, p. 256 et seq.

CHAPTER IV.

PROBLEMS ON THE EQUILIBRIUM OF A SOLID BODY.

PROBLEM I.—(59.) A bent rod or lever ACB is suspended at C, about which point it is free to move in a vertical plane, and weights P, P_3 are attached to its extremities: to find the position in which it will rest (fig. 47).

Let us take the fixed point C for the centre of moments, then it will be sufficient for the equilibrium that the moments of the applied forces balance each other. These applied forces are first, the weight P_1 of the arm CA acting at its centre of gravity, the middle point a; secondly, the weight P acting at A; thirdly, the weight P_2 of the arm CB acting at the middle point b; and lastly, the weight P_3 acting at B: these forces all have vertical directions, and the moments of the two latter oppose the moments of the two former. Hence, drawing the perpendiculars from C on the directions of the forces as in the figure, we have this equation for the conditions of equilibrium, viz.

$$P \cdot Cp + P_1 \cdot Cp_1 = P_2 \cdot Cp_2 + P_3 \cdot Cp_3,$$

and it is from this equation that the required position, that is, the angle ACp or the angle BCp_3, must be determined. Put α for the angle ACp, α' for the angle BCp_3 and θ for the given angle ACB; also, call the given length AC, $2a$, and the given length BC, $2a'$, then

$$Cp = 2a \cos. \alpha,\ Cp_1 = a \cos. \alpha,\ Cp_2 = a' \cos. \alpha' = a' \cos. (\alpha+\theta),$$
$$Cp_3 = 2a' \cos. \alpha' = -2a' \cos. (\alpha+\theta);$$

hence the equation of equilibrium is the same as

$$(2P+P_1)\, a \cos. \alpha = -(P_2+2P_3)\, a' \cos. (\alpha+\theta),$$

where α is the only unknown quantity. Or, since

$$-\frac{\cos. (\alpha+\theta)}{\cos. \alpha} = \frac{\sin. \alpha \sin. \theta - \cos. \alpha \cos. \theta}{\cos. \alpha} = \tan. \alpha \sin. \theta - \cos. \theta,$$

the equation reduces to

$$(2P+P_1)\, a = P_2+2P_3)\, a' (\tan. \alpha \sin. \theta - \cos. \theta)$$
$$\therefore \tan. \alpha = \frac{(2P+P_1)\, a + (P_2+2P_3)\, a' \cos. \theta}{(P_2+2P_3)\, a' \sin. \theta}.$$

If the extremities of the lever carry no weights

$$\tan. \alpha = \frac{P_1\, a + P_2\, a' \cos. \theta}{P_2\, a' \sin. \theta}.$$

These results would remain unaltered, though the two arms CA, CB were of different thickness; but when they are equally thick, since their weights must then be as their lengths. or, since $\frac{P_2}{P_1} = \frac{a'}{a}$,

the last expression may be put under the form

$$\tan.\ a = \frac{a^2 + a'^2 \cos.\ \theta}{a'^2 \sin.\ \theta}.$$

PROBLEM II.—(60.) An oblique cylinder stands on a horizontal plane, its inclination to which is 60°, perpendicular height 4 feet, and diameter of the base 3 feet. Required the diameter of the greatest sphere of the same material as the cylinder that will hang suspended from the upper edge (fig. 48,) without overturning the cylinder.

The centre of gravity G of the cylinder is at the middle of its axis CD; hence the acting forces are the weight of the cylinder in the vertical direction GE, and the weight of the sphere in the vertical direction B'P and these two forces are just sufficient to prevent any tendency to motion about B; hence the equation of equilibrium is

$$\text{BE} \times \text{cylinder} = \text{BF} \times \text{sphere}.$$

Now by trigonometry sin. GDE : DGE : : GE : DE $=2\sqrt{\tfrac{1}{3}}$

$$\therefore \text{BF} = 2\text{DE} = 4\sqrt{\tfrac{1}{3}},\ \text{BE} = \text{BD} - \text{DE} = \tfrac{3}{2} - 4\sqrt{\tfrac{1}{3}};$$

hence, substituting the volumes of the cylinder and sphere for their weights to which they are proportional, the equation of equilibrium is

$$(\tfrac{3}{2} - 4\sqrt{\tfrac{1}{3}}) \times 3^2 \times \cdot 7854 \times 4 = 4\sqrt{\tfrac{1}{3}} \times \tfrac{2}{3} \times \cdot 7854 \times \overline{\text{diameter}}\,\rceil^3$$

$$\therefore \text{diameter} = 2 \cdot 006 \text{ feet.}$$

PROBLEM III.—(61.) One extremity C of a heavy rod is moveable about a fixed point in a vertical plane (fig. 49), and to the other extremity B is fastened a cord which goes over a pulley A in a horizontal line with C, and supports a weight P equal to half the weight of the rod: required the position in which the rod will rest.

The forces which prevent motion about C are the tension of the cord BA, measured by the weight P, and the weight 2P of the rod acting vertically at the middle of CB.

Draw BF perpendicular to AC, and CE perpendicular to AB, then the equation of equilibrium is

$$\text{P} \cdot \text{CE} = 2\text{P} \cdot \tfrac{1}{2}\text{CF} = \text{P} \cdot \text{CF} \therefore \text{CE} = \text{CF}.$$

Put BC$=a$, AC$=b$, and AF$=x$, then the right-angled triangles AEC, AFB being similar, we have

$$\frac{\text{EC}^2}{\text{AC}^2} = \frac{\text{BF}^2}{\text{BA}^2} \therefore \frac{(b-x)^2}{b^2} = \frac{a^2 - (b-x)^2}{x^2 + a^2 - (b-x)^2} = \frac{a^2 - b^2 + 2bx - x^2}{a^2 - b^2 + 2bx}$$

an equation which reduces to

$$2bx^3 - (4b^2 - a^2)\,x^2 - 2b\,(a^2 - b)\,x = 0.$$

One root of this equation is $x=0$, and the other two, as given by the quadratic, $x^2 - \frac{4b^2 - a^2}{2b}x = a^2 - b^2$. (1)

are $x=b-\frac{a}{4b}(a\pm\sqrt{\{8b^2+a^2)\}}$ . . . . (2).

Let us examine into the nature of these as connected with the positions of the rod. The first root $x=0$ gives the position in fig. 50, which position, however, the rod cannot take if AC is greater than CB, that is if b exceeds a; but when b does not exceed a, then this will be one of the positions of equilibrium, and it may be remarked that whatever be the weight of the rod moveable about C, the other end B will always be supported in every position by a vertical force equal to half that weight.

There cannot be any other position of equilibrium for the same weight P between the lines CB, CA, because in no such position can the condition of equilibrium, viz. CE=CF, have place; hence no negative root of (1) can be consistent with the conditions of the question.

To determine in what circumstances the two roots are admissible, put $a=nb$, then the values of x will be

$$x=b\,(1-\frac{n^2\pm n\sqrt{\{8+n^2\}}}{4}),$$

which for $n=1$ gives $x=0$, and $x=\frac{3}{2}b$; when n is less than 1, then it is obvious that both values of x are positive; hence, when a is not greater than b, there are always two positions of equilibrium determinable from the two roots or values of x in (2). As one of these values of x is always greater than b, the corresponding position of the rod will be as in fig. 51. When n is greater than 1, one of the above values of x is always positive and the other always negative; hence, when a is greater than b, there is but one position of equilibrium (fig. 52) determinable from the positive root or value of x in (2), but then there is another position determinable from $x=0$, that is, the extremity B will rest in the vertical line from the pulley, as in fig. 50.

PROBLEM IV.—(62.) AD and BC (fig. 53) are two heavy bars moveable in a vertical plane about their extremities A, B in the horizontal line AB; required the position in which they will rest by leaning against each other.

Call the weight of the bar AD, acting vertically at its middle E, P; and the weight of the bar BC, acting vertically at G, P_1; then the forces acting on AD, to turn it round A, are P in the direction EK, and the pressure of CB in the direction DL perpendicular to BC; call this pressure P_2, then the bar AD being at rest, we must have $P.AK=P_2.AL$, also the forces acting on BC, to turn it round B, are P_1 in the direction GH, and the pressure P_2 in the perpendicular direction LD; hence, $P_1.BH=P_2.BD$, these two are therefore the equations of equilibrium. Eliminating P_2, we have the single equation

$$P \cdot AK \cdot BD = P_1 \cdot BH \cdot AL.$$

Put $AB=a$, $AD=b$, $BC=c$, and $BD=x$; then

$$AK = AE \cdot \cos. A = \frac{\frac{1}{2} b\,(b^2+a^2-x^2)}{2\,ab}$$

$$BH = BG \cdot \cos. B = \frac{\frac{1}{2} c\,(a^2+x^2-b^2)}{2\,ax}$$

$$AL = AD \cdot \cos. D = \frac{b^2+x^2-a^2}{2\,x}.$$

Hence, by substitution, we have

$$2\,b\,(b^2+a^2-x^2)\,P = bc\,(a^2+x^2-b^2)\,(b^2+x^2-a^2)\,P_1;$$

an equation of the fifth degree, from which, when numbers are put for a, b, and c, the values of x may be determined, (see Algebra, p. 210.)

PROBLEM V.—(63.) A given rod or beam, not of uniform thickness, has one end suspended by a cord of a given length, fixed at a given point above an inclined plane of a given inclination, and the other end of the beam is sustained by the inclined plane; it is required to determine the position of the beam, weight sustained by the cord, and pressure against the inclined plane when the beam is at rest (fig. 54).

Let P be the given point, AP the string, AB the position of the beam when at rest, with its end B on the given inclined plane BK.

The forces acting on the beam are the tension of the string in the direction AP, the weight of the beam acting vertically at the given point G, its centre of gravity, and the pressure at B acting in the given direction BD′, perpendicular to the plane. Let PA, D′B, be produced till they meet in D, then, as the resultant of the forces acting in these lines must pass through D, and as the direction of this resultant is vertical, it follows that the vertical line DG must pass through the centre of gravity. Hence to determine the position of the rod, draw AH, PK, parallel to D′D, and AM parallel to BK, also produce BA to N; then the position will be ascertained, if we can find the angle $PCK = PAM = \theta$. The known quantities are $BG=a$, $GA=b$, $PK=c$, $PA=l$, $CBQ=i$=inclination of the plane; if, therefore, we call the unknown angle ABK, or NAM, ϕ, we shall have

$$PM = l \sin. \theta, \quad AH = KM = (a+b) \sin. \phi$$

$$\therefore\ PM - KM = l \sin. \theta + (a+b) \sin. \phi = c \quad . \quad . \quad . \quad . \quad (A).$$

If, therefore, we can obtain another equation involving no other unknown quantities besides θ and ϕ, this latter may be eliminated, and thence θ determined. Now two different expressions for BD, involving only these unknown quantities, may readily be obtained

from the two triangles ABD, GBD, which have this side in common; for observing that

$$\text{PAN}=\text{BAD}=\theta-\phi$$
$$\text{ADB}=\text{APM}=90^\circ-\theta$$
$$\text{BDG}=\text{KBQ}=i$$
$$\text{BGD}=\text{comp. QBG}=90^\circ-(i+\phi);$$

we have, from the triangle ABD,

$$\sin.\ \text{ADB} : \sin.\ \text{BAD} :: \text{AB} : \text{BD};$$

that is, $\cos.\ \theta : \sin.\ (\theta-\phi) :: a+b : \text{BD}$ . . . . (1);

and in the triangle BGD, we have

$$\sin.\ \text{GDB} : \sin.\ \text{DGB} :: \text{BG} : \text{BD};$$

that is, $\sin.\ i : \cos.\ (i+\phi) :: a : \text{BD}$ . . . . (2).

Equating now the two expressions for BD, furnished by (1) and (2), we have $\dfrac{(a+b)\ \sin.\ (\theta-\phi)}{\cos.\ \theta}=\dfrac{a\ \cos.\ (i+\phi)}{\sin.\ i}$

$\therefore\ (a+b)\ \sin.\ i\ \sin.\ (\theta-\phi)=a\ \cos.\ \theta\ \cos.\ (i+\phi)$ . . . . (B);

hence, by means of equations (A) and (B), θ may be determined, and thus the position of the beam found.

It remains to determine the tension T of the string, and the pressure P on the inclined plane. In order to this draw GV parallel to PA, then, since the sides of the triangle GVD are respectively parallel to the three forces, they, or the sines of their opposite angles, are proportional to these forces; that is, calling the force in GD, or weight of the beam W, we have

$$\sin.\ \text{GVD} : \sin.\ \text{VDG}=\sin.\ i :: \text{W} : \text{T}$$
$$\sin.\ \text{GVD} : \sin.\ \text{VGD}=\cos.\ \theta :: \text{W} : \text{P};$$

consequently, $\text{T}=\dfrac{\sin.\ i}{\cos.\ \theta}\text{W},\ \text{P}=\dfrac{\cos.\ (\theta+i)}{\cos.\ \theta}\text{W}.$

Problem VI.—(64.) A given beam AB is supported by strings which go over pullies C, D, and have given weights P, P_1, attached to them, to find the position of equilibrium (fig. 55).

Produce the strings till they meet in g, then the vertical gE will pass through the centre of gravity G of the beam, and if GF be drawn parallel to Dg the three sides of the triangle GFg will be proportional to the three equilibrating forces, and these are all given. Put AG$=a$, GB$=b$, the inclination of CD to the horizontal line CK, or DK′, i; these quantities are also given. Let α represent the angle DCA, and β the angle CDg, then sin. $\text{BDP}_1=\cos.\ (\beta+i)=\sin.\ \text{FG}g$, and $\cos.\ g\text{CK}=\cos.\ (\alpha-i)=\sin.\ \text{F}g\text{G}$;

hence, from the triangle GFg, we have $\dfrac{\text{P}}{\text{P}_1}=\dfrac{\cos.\ (\beta+i)}{\cos.\ (\alpha-i)}$ . . . (1)

$$\frac{\text{P}}{\text{W}}=\frac{\cos.\ (\beta+i)}{\sin.\ (180^\circ-\beta-\alpha)}=\frac{\cos.\ (\beta+i)}{\sin.\ (\beta+\alpha)}.$$

From these two equations the unknown angles α and β may be determined. But to find the position of AB, we must also know the angle $A'=\delta$. From G draw the perpendiculars Gp, Gp_1, on the strings, then $Gp=GA \sin. A=a \sin. (\alpha-\delta)$, and

$$Gp_1=GB \sin. GBp_1=b \sin. (\beta+\delta),$$

hence
$$a \sin. (\alpha-\delta)\, P=b \sin. (\beta+\delta)\, P_1$$

$$\therefore \frac{P}{P_1}=\frac{b \sin. (\beta+\delta)}{a \sin. (\alpha-\delta)} \quad \ldots . (2);$$

from which equation δ may be determined, α and β having been previously found. The quantities thus found enable us to find the two unknown sides of the triangle CgD, and those of the triangle AgB, and thence the distances CA, DB.

Problem VII.—(65.) A given beam AB hangs by two strings, CA, DB, of given lengths, from two given fixed points C, D; to find the position in which it will rest (fig. 56).

Here instead of the tensions we have the lengths a', b' of the strings CA, DB; hence, using the same notation and the same reasoning as in the last problem, we get two different expressions (1), (2), for the ratio of these tensions; that is, we have one equation between the unknowns α, β, and δ; it will, therefore, require two more equations to determine them; these two may be readily obtained, since we may deduce two different expressions for the perpendiculars from A and B on the strings. Thus, if we draw AM parallel to BD, it is plain that the perpendicular from A on DB will be equal to that from C on DB, *minus* that from C on AM; that is, calling CD, c, the expression for this perpendicular will be

$$c \sin. \beta-a' \sin. (\alpha+\beta);$$

but the expression for the same perpendicular is also AB sin. B, that is
$$(a+b) \sin. (\beta+\delta)$$

$$\therefore c \sin. \beta-a' \sin. (\alpha+\beta)=(a+\beta) \sin. (\beta+\delta) \quad \ldots . (1).$$

Again the perpendicular from B on Ag is equal to the perpendicular on it from D, *minus* that from D on BN parallel to Ag, the expression for this perpendicular is, therefore,

$$c \sin. \alpha-b' \sin. (\alpha+\beta);$$

but the expression for the same perpendicular is also AB sin. A, that is,
$$(a+b) \sin. (\alpha-\delta)$$

$$\therefore c \sin. \alpha-b' \sin. (a+\beta)=(a+b) \sin. (\alpha-\delta) \quad \ldots (2).$$

These equations, in conjunction with that before mentioned, viz.

with
$$\frac{\cos. (\beta+i)}{\sin. (\beta+\alpha)}=\frac{b \sin. (\beta+\delta)}{a \sin. (\alpha-\delta)} \quad \ldots . (3)$$

are sufficient to determine the unknown quantities sought.

Problem VIII.—(66.) A given solid hemisphere, with its convex part upon a smooth inclined plane of given inclination, is kept

from sliding by a string of given length having one end fastened to a given point, and the other end attached to the edge of the hemisphere: it is required to determine the point where the hemisphere touches the plane when at rest, the pressure on the plane, and the tension of the string (fig. 57).

Let P be the given point, PI the string, IFM the hemisphere, C its centre, and CT perpendicular to MI, then the distance CG of its centre of gravity is $\frac{3}{8}$ CT. The forces acting on the body are the weight W in the vertical direction GR, the pressure P in the direction FO perpendicular to the inclined plane AB, and the tension T in the direction IP: this last direction must when produced meet in O the point of concurrence of the other two forces. To determine the point F, draw GN, IL, and PE, perpendicular to CF, ID perpendicular to PE, and PH to AB, then the given quantities are

$$CG=a,\ CF=b,\ PH=EF=c,\ PI=l,\ \text{cot. angle } BAR=\text{cot. angle } GON=t;$$

and we wish to find HF or PD and IL. Put x and y for the sine and cosine of the angle GCN, or CIL, then we have

$$CN=ay,\ GN=ax,\ CL=bx,\ LI=by$$

$$\therefore\ ON=t\cdot GN=tax,\ CO=CN-ON=a(y-tx)$$

$$LO=CL-CO=bx-a(y-tx),\ LE=CE-CL=ID=b-c-bx.$$ By the similar triangles OLI, IDP,

$$OL:LI::ID:DP\ \therefore\ DP=\frac{by(b-c-bx)}{bx-a(y-tx)};$$

and since $DI^2+DP^2=IP^2=l^2$

$$\therefore\ (b-c-bx)^2+\left\{\frac{by(b-c-bx)}{bx-a(y-tx)}\right\}^2=l^2.$$

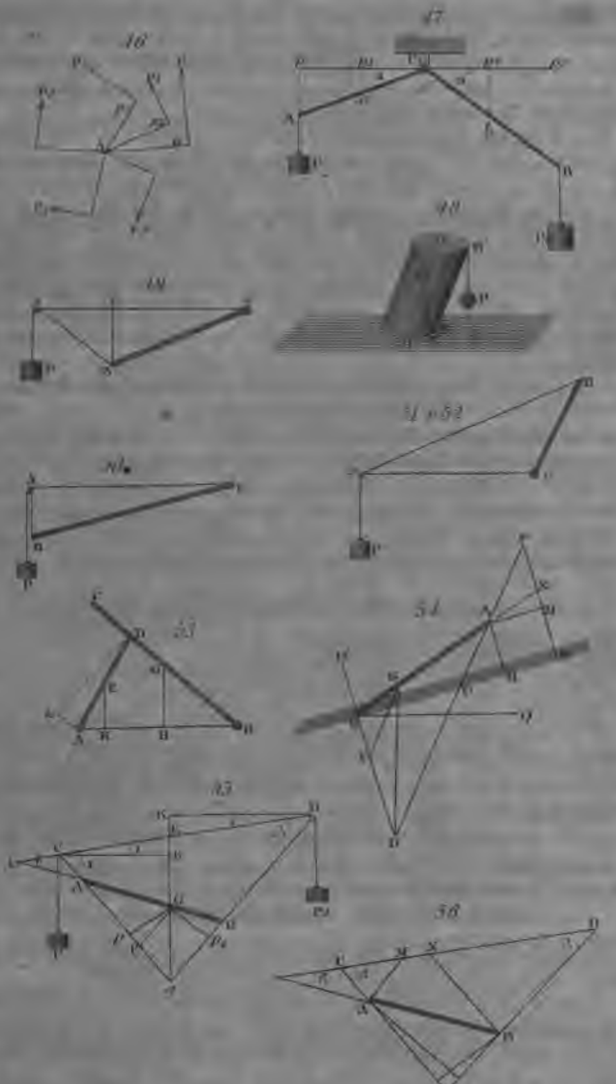
This equation joined to $x^2+y^2=1$ is sufficient to determine x and y, and thence DP and LI. The sides LI, LO, being now known, the angle LOI is known; hence, taking the centre of moments at L we have

$$OL\ \text{sin. } LOI\times T=OL\ \text{sin. } LOG\cdot W;\ \text{and } LOG=\angle A,$$

$$\therefore\ T=\frac{\text{sin. } LOG}{\text{sin. } LOI}W;\ \text{also, } P=T\ \text{cos. } LOI+W\ \text{cos. } LOG$$
$$=(\text{sin. } LOG\cdot\text{cot. } LOI+\text{cos. } LOG)\,W.$$

PROBLEM IX.—(67.) To determine the position in which a paraboloid ABC will rest upon a horizontal plane (fig. 58).

Suppose P to be the point on which it rests, then the pressure being in the vertical direction PG, is in the normal to the surface at P, and, moreover, passes through the centre of gravity G. Taking the axis AX, AY the equation of the vertical section, through AX and PG is $y^2=2px$; therefore, the subnormal NG is $NG=y\frac{dy}{dx}=$



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p; but, calling the altitude AX of the paraboloid a, the distance AG is (ex. 3, p. 71) $AG=\frac{2}{3}a$, $\therefore x+p=\frac{2}{3}a \therefore x=\frac{2}{3}a-p$;
this therefore is the abscissa of the point on which the body rests, hence, the tangent of the inclination XRP of the axis to the horizon, that is, the value of $\frac{dy}{dx}$, for the point P is

$$\tan.\theta=\frac{dy}{dx}=\sqrt{\frac{h}{2x}}=\sqrt{\{p+2(\tfrac{2}{3}a-p)\}}\ (1).$$

Besides the point thus determined, there is no other on which we can place the body where the normal shall be equal to the distance of the centre of gravity from the horizontal plane, which equality must exist in order that the body may rest on that point, making, however, one exception, viz. when the point is the vertex A; for at this point the normal being in the line of direction of the centre of gravity, the body would necessarily rest on it, be the length of the normal what it may. In order that the body may be unable to rest on any other point besides this, the altitude must be such that between the vertex and base there shall be no point whose normal shall equal the distance of the centre of gravity from the plane, therefore the condition is that the result (1) may be either impossible or infinite, which requires that

$$a<\frac{3}{2}p,\ \text{or}\ a=\frac{3}{2}p;$$

when, therefore, $a=\frac{3}{2}p$ we see that the vertex is the point at which the normal measures its distance from the centre of gravity. Hence if any segment of a paraboloid, whose altitude does not exceed $\frac{3}{2}p$, be placed any how on a horizontal plane, except indeed on its base, it must always restore itself to an upright position and rest, when it does rest, on its vertex.

PROBLEM X.—(68.) To determine the pressures exerted by a door on its hinges or on the two pivots upon which it hangs.

Call the weight of the door acting at its centre of gravity P the distance of the centre of gravity, that is, of the direction of the force from the vertical axis x, and the distance between the pivots $z'-z''$; then there being no pressures in the direction of the axis of y, the equations 1 at page 81, become, in this case,

$$X'+X''=-P\cos.\alpha=0,\ Z'+Z''=-P\cos.\gamma=-P\ldots(1)$$
$$\therefore X'=-X'';$$

hence the equations (2) furnish only $(z'-z'')X'=-Px$

$$\therefore X'=\frac{Px}{z''-z'}\ \therefore X''=\frac{Px}{z'-z''}\ldots(2).$$

The equation (1) shows that the door presses with its whole weight

P, the vertical line of the hinges; and the equations (2) express the horizontal forces on the hinges, the lower hinge being pushed inwards, and the upper hinge drawn outwards, each with a force equal to $\frac{Px}{z'-z''}$.

PROBLEM XI.—(69.) One end of a uniform beam of weight W is moveable round a fixed point in a vertical plane, and to the other end is attached a string which passes over a pulley, and is loaded with a given weight P; the fixed point and pulley are in a horizontal line, and their distance asunder is equal to the length of the beam, which, however, is not given; required the angle θ between the beam and horizontal line when the beam is at rest,

$$\cos.\,\theta=\frac{P}{2W}\pm\left(\frac{P^2}{4W^2}+\frac{1}{2}\right)^{\frac{1}{2}}.$$

PROBLEM XII.—(70.) A cone of marble, the axis of which is twenty feet, and base diameter, six feet, stands on the edge of its base, the axis making an angle of 60° with the plane of the horizon; what must be the direction and intensity of the least force applied to its vertex that will just sustain the cone in that position; the weight of the cone being 284 cwt.?

The least force is 1·377 cwt. acting perpendicular to the lower side of the cone.

PROBLEM XIII.—(71.) What will be the height of the greatest segment that can be cut off a prolate spheroid whose longer axis is double the shorter, by a plane perpendicular to the longer axis, so that it may be unable to rest upon any point of its convex surface except the vertex.?

Height = semi. trans. $\times$ $(3-\sqrt{5})$.

PROBLEM XIV.—(72.) A beam of given weight W, rests with one end against a vertical wall and the other upon an inclined plane; calling the inclination of the beam to the horizon i, the pressure against the wall P, and the thrust against the inclined plane T, to determine the intensities of P and T.

$$P=\frac{W}{2\tan.\,i};\;T=\frac{\sqrt{\{\sin.^2 i+\frac{1}{4}\cos.^2 i\}}}{\sin.\,i}W.$$

PROBLEM XV.—(73) A ladder of uniform thickness and weight w pounds, is placed against a vertical wall, and at a given inclination i to the horizon, a person weighing W pounds ascends the ladder, and it is required to determine the pressure at the top and the thrust at the bottom of the ladder when the person arrives at any given height.

CHAPTER IV.

ON THE MECHANICAL POWERS.

(74.) HAVING now considered pretty fully the theory of the equilibrium of forces, applied to different points of a solid body, it is proper that we should speak of those cases in which the forces are not immediately applied to the body, on which their influence is ultimately exerted, but to some intermediate body contrived for the purpose of transmitting this influence in the most advantageous manner. These contrivances are called *Machines*, and the simple elements or constituent parts of all machinery are called the *mechanical powers*. These are six in number, and are as follow: the *Lever*, the *Wheel and axle*, the *Pulley*, the *Inclined Plane*, the *Screw*, and the *Wedge*.

The Lever

(75.) A lever is a rigid bar or rod, moveable about a fixed point or *fulcrum*, and it is divided into three different kinds, depending on the position of this fulcrum with respect to the applied force, and the body to be influenced by it; thus, if the fulcrum be between the force or power and the body or weight, as in fig. 59, the lever is said to be of the first kind; if it be situated so that the weight is on the middle, as in fig. 60, the lever is of the second kind, and when, as in fig. 61, the power is in the middle, the lever is of the third kind. In the figures the rods are straight, but they would still be levers if bent or curved.

That a lever acted upon by any forces may be in a state of equilibrium, it is, obviously, merely necessary that the sum of the moments, taking the fulcrum as the centre, be equal to 0; this condition, therefore, comprises the whole theory of the lever, so that when, as is commonly the case, the equilibrating forces are a weight W and a power P, then the condition is $W p_1 = Pp \therefore \frac{W}{P} = \frac{p}{p_1}$ (1); hence the weight and power are to each other, reciprocally, as their distances from the fulcrum. If, as is usually the case, the object be to balance W with the least power possible, this must act so that p may be the greatest possible, and, therefore, the power must act perpendicularly to the lever.

When the weight w of the lever itself is to be taken into consideration we must view it as a third force acting at its centre of gra-

vity ; if we call the distance of this new force from the fulcrum g, then the above equation will be $Wp_1 \pm wg = Pp$. (2); the upper or lower sign being used according as this force tends to favour or to oppose W.

(76.) The mechanical advantage of the single lever may be considerably increased by combining several together, so that the power of one may be communicated to another.

Thus in the system of levers, represented in fig. 62, if we call the arms which are on the same side of the fulcra as the power p, p_1, p_2, &c. and the other arms p', p'_1, p'_2, &c. and, moreover, call the powers acting at the extremities of the latter arms P_1, P_2, &c. then in the case of equilibrium we shall have the equations

$$Pp = P_1 p',\ P_1 p_1 = P_2 p_1',\ P_2 p_2 = P_3 p_2',\ \&c.$$

and multiplying these together,

$$P p p_1 p_2 \cdots = P_n p' p'_1 p'_2 \cdots ;$$

or, the last power P_n being the weight W,

$$P p p_1 p_2 \cdots = W p' p'_1 p'_2 \cdots ;$$

so that the power is to the weight, as the product of the arms on the side of the weight, to the product of the arms on the side of the power.

PROBLEM I.—(77.) A body is weighed successively in the two scales of a false balance, in the one scale it balances a weight p, in the other a weight q, required the true weight of the body.

Suppose the lengths of the arms of the balance to be a and b, and let x represent the true weight of the body, then its moment in one scale is ax, and in the other bx, and, by the problem,

$$ax = bp,\ bx = aq ;$$

multiplying these together we have, $abx^2 = abpq \therefore x = \sqrt{pq}$; hence the true weight is a mean proportional between the two false weights.

PROBLEM II.—(78.) The common steelyard (fig. 63) is a bar AB, moveable about a fulcrum O ; P, the body to be weighed, is hung at the shorter arm A, and a given weight W is moved along the other arm till it balances P ; then the weight of P is known from the place of W ; to find how the distances OC increase with regard to the weight P.

Let G be the centre of gravity of AB, and Gg vertical meeting the horizontal line AOC in g. Call the weight of AB, w, $OA = p$, $OC = p_1$, $Og = g$, then, by equation (2) above,

$$W p_1 + wg = Pp \therefore p_1 + \frac{wg}{W} = \frac{Pp}{W} ;$$

hence, if we take $Ox = \frac{wg}{W}$, we have $xC = \frac{Pp}{W}$; hence xC varies as P; and, proceeding from x, equal additions of distance to xC correspond to equal additions of weight to P: that is, if xC be graduated with points 1, 2, 3, &c. at equal successive intervals from x, W placed at these points will successively balance the weights 1, 2, 3, &c.

PROBLEM III.—(79.) A homogeneous lever AB (fig. 64) of the second kind is equally thick throughout; it is required to determine what must be the length of the arm AB, that a given weight W, acting at the extremity of the arm AC, may be supported by the least power P possible, taking into account the weight of the lever itself.

The lever being homogeneous, and equally thick throughout any portion of its length, may be taken to represent the weight of that portion; hence, calling the length AB of the lever x, its weight will be x, acting at G, its middle point; therefore, putting $AC = p$, we have, by equation (2),

$$Wp_1 + \tfrac{1}{2}x \cdot x = Px \therefore P = \frac{Wp_1}{x} + \tfrac{1}{2}x.$$

To determine for what value of x this expression for P is a minimum, we have $\frac{dP}{dx} = -\frac{Wp_1}{x^2} + \frac{1}{2} = 0$; from which we immediately get $x^2 = 2Wp_1 \therefore x = \surd\{2Wp_1\}$, the value required.

It must be remembered that we have taken lengths to represent weights, so that, whatever our unit of length has been, the weight of that unit must be considered as the unit of weight, thus if we have measured p_1, and therefore x, in inches, then the numerical expression for W will be the weight W, divided by the weight of an inch of the lever.

By substituting this value of x in the foregoing expression for P, we have, for the intensity of this power when acting at the greatest advantage,

$$P = \frac{Wp_1}{\surd 2Wp_1} + \tfrac{1}{2}\surd\{2Wp_1\} = \surd\{2Wp_1\}.$$

The Wheel and Axle.

(80.) This machine is in reality only a modification of the lever; it consists of two parts, a cylinder called the *axle*, and the surrounding circle or *wheel* connected with it, having its centre in the axis of the cylinder about which the whole turns (see fig. 65).

This machine is not complete in itself like the lever, requiring the addition of cords, or chains, or some other intermediate body, to communicate with the forces engaged.

The more immediate object of this machine is to support or raise a weight W, suspended to a rope wound about the axle, by means of a power P applied to the circumference of the wheel. But in numberless applications of this machine the axle does not communicate directly with the weight by a cord: but, by being surrounded with teeth, as in fig. 66, acts upon a toothed wheel, and the axle of this last upon another toothed wheel, and so on, the power being thus transmitted to the weight W. The effect of a power thus transmitted we shall consider presently, examining first the more simple case just noticed.

Let the radius of the wheel be p, that of the axle p_1, then (72)

$$W p_1 = Pp \therefore \frac{W}{P} = \frac{p}{p_1};$$

the weight and power being to each other, reciprocally, as their distances from the fixed axis, as in the lever.

If the equilibrating forces do not act tangentially, as we here suppose, then instead of p, p_1 representing the radii, they will represent the distances of the directions of the power and weight from the axis.

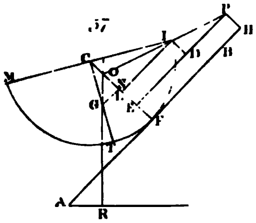
It is obvious that the greater be the wheel the less will be the power requisite to support or move a given weight: and that a continually decreasing power may have a uniform effect upon a constant weight, it must act upon a series of continually increasing wheels, constantly keeping up the above proportion, the radii of the wheels varying inversely as the powers: thus P and P′ being any two values of the powers we have

$$\frac{W}{P} = \frac{p}{p_1}, \frac{W}{p_1} = \frac{p'}{p_1} \therefore \frac{p}{p'} = \frac{P'}{P}.$$

In this way the varying power exerted by the main-spring of a watch while uncoiling is made to produce a uniform effect; this power acting on a series of varying wheels (fig. 67) called the *fuzee*.

It should be remarked that the machine we are now considering is virtually unchanged, though the wheel be stripped of its rim and the power be applied at the extremities of the spokes.

It should be further remarked that the thickness of the rope, when considerable, must not be neglected in estimating the conditions of equilibrium; for we ought to consider the forces to be transmitted along the middle or axis of the rope, and, therefore, the radius of the rope should be added to that of the wheel, and of the axle, so that in the above expressions p and p_1 are the distances of the axes of the ropes from the axis of the machine.



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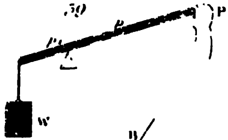

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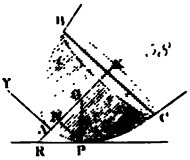

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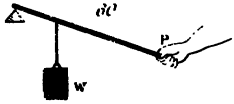

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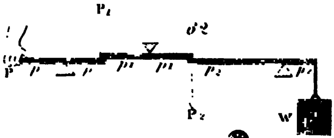

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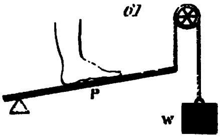

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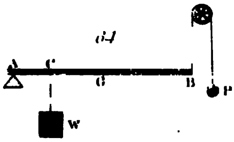

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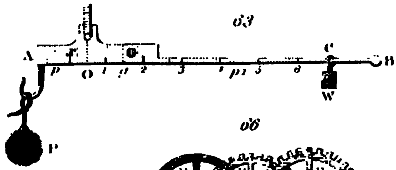

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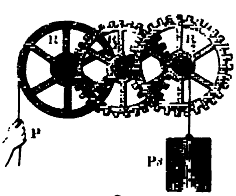

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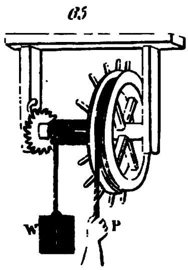

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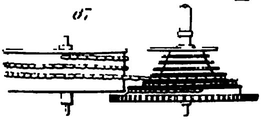
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(81.) When the axle is toothed it is called a *pinion*, and the teeth its *leaves*. If a power be in equilibrium with a weight by means of a system of toothed wheels and pinions, as in fig. 66, then we shall find that the power will be to the weight as the product of the radii of the pinions to the product of the radii of the wheels.

For let the radii of the axle or pinions be r, r_1, r_2, &c. and those of the wheels R, R_1, R_2, &c. then the power P acting on the first wheel equilibrates a weight or power P_1, on the pinion, expressed by $P_1 = \frac{PR}{r}$; this then is the power applied to the second wheel and which, therefore, communicates to its pinion a power expressed by

$$P_2 = \frac{P_1 R_1}{r_1} = \frac{P R R_1}{r r_1};$$

this is the power applied to the third wheel, and continuing this computation it is plain that the power which the nth pinion or axle acts is $P_n = \frac{P R R_1 R_2 \dots R_{n-1}}{r r_1 r_2 \dots r_{n-1}}$; and consequently, P_n is the weight which the power P will balance on the nth axle.

Instead of expressing this result in words, as above, we may say, since the radii are as the circumference, and these, again, as the number of teeth they carry, that the power is to the weight as the product of the numbers expressing the leaves to each pinion to the product of the numbers expressing the teeth to each wheel; the number for the first wheel, which is plane, expressing the number of teeth it *could* carry, and, in like manner, the number for the last axle being that expressing the number of leaves it would carry.

For further particulars respecting *tooth and pinion work*, and, indeed, respecting machinery in general, the student is referred to Professor Gregory's Treatise of Mechanics, a work abounding with valuable information. Much interesting matter will also be found in Dr. Lardner's elegant volume on Mechanics, in the Cabinet Cyclopædia.

SCHOLIUM.

(82.) The student will have observed that the foregoing theory of toothed wheels is founded on the supposition that the power communicated from the tooth of one wheel to that of another is in the direction of a tangent to the circle on which this latter is raised, as in the plane wheel and axle, and that such may really be the direction of the power, a particular figure must be given to the teeth, at least to the working sides of them. This figure is that of the involute of the circle on which they are raised. Thus, let IHF, KEb

(fig. 68) be the wheels to which the teeth are to be accommodated, the acting face GCH of the tooth *a* must have the form of the curve traced by the extremity H of the flexible line F*a*H, as it is unwrapped from the circumference; and, in like manner, the acting face of the tooth *b* must be formed by the unwrapping of a thread from the circumference of the circle KE*b*. The line FCE drawn to touch both circles will cut the surfaces of the two teeth in C, the point where they touch each other, at a point in the common tangent to both circles, and the force arising from their mutual pressure will always act in the direction of the circumference of the wheels at E and F. But, continues Dr. Gregory, whose words we have borrowed in the preceding description, although Roemer, Varignon, De la Hire, Camus, Euler, Emerson, Kaestner, and Robison, have turned their thoughts to this object, and some of them have struck out rules of ready application in practice, it is to be regretted that these rules have been little followed by practical mechanics, most of whom have, in this case, been more inclined to follow a set of hackneed rules handed down from one workman to another, although completely destitute of scientific principle. Even watchmakers, in whose constructions a little more than common skill and nicety in the execution might be expected, are but few of them acquainted with any rules founded upon the deductions of accurate theory; but commonly, we are informed, give to their teeth the shape assumed by a horse hair when held bent between the fingers, a method so vague that it is difficult to conceive how it came to be adopted.

The Pulley.

(83.) A pulley is a grooved wheel moving freely on an axis, and fixed in a case or *block*. It communicates applied force in conjunction with a cord which the groove receives.

The *fixed pulley* we have already employed in various parts of this work for the sole purpose to which it is applicable, viz. for the purpose of changing the direction of a force acting by a cord, and although the fixed pulley is, for this purpose even, an important instrument, yet as it does not afford any mechanical advantage in the way of accumulating force it offers no theory for discussion. It is different with the moveable pulley (fig. 69,) to the block of which the weight is fastened, which is sustained between the power P, acting at one end of the cord, and the pressure on a fixed hook Q at the other end; the tension of the cord being uniform, it is obvious that the intensity of P must be equal to the strain or pressure on Q. Let us examine the relation which these equal powers bear to the weight. Continue the directions of the ropes PC, QD, till they meet, unless they should be parallel; then, since the system of

forces P, Q, W is in equilibrium, the point E, where the directions of two meet, must be in the direction of the third, and the resultant W of the equal forces P, Q will be

$$W=2\,P\cos.\alpha \;.\;.\;.\;.\; (1);$$

α being equal to half the inclination of the cords from P and Q. Instead of the tabular cosine we may introduce into this expression the cosine corresponding to the radius r of the pulley, provided we write $\frac{\cos.\alpha}{r}$ instead of $\cos.\alpha$; this cosine is, obviously, the line Cn, being the sine of the angle COn the complement of α; hence, dividing each side of the expression thus changed by P, we have

$$\frac{W}{P}=\frac{2\cos.\alpha}{r} \;.\;.\;.\;.\; (2);$$

that is, the power is to the weight as the radius of the pulley to the chord of that arc of it which is in contact with the rope.

If the cords from P and Q are parallel, then the equilibrium being the effect of three parallel forces, the middle force W must be equal to the two P and Q; in this case, therefore, $W=2\,P$. (3); and it is obvious that half the circumference of the pulley must be in contact with the rope.

It may be observed from the expression (1) that the moveable pulley affords a mechanical advantage only so long as the inclination of the cords from P and Q is less than 120°; for at this angle $\cos.\alpha=\frac{1}{2}$, and at greater angles $\cos.\alpha$ is less than $\frac{1}{2}$. The greatest advantage is gained when the cords are parallel.

(84.) The advantage gained by a single moveable pulley may be multiplied to any extent by employing a *system of pullies*, as in fig. 70, thus, representing the tensions of the several ropes by the letters annexed to them in the figure, we have from equation (1) above

$$\begin{aligned} W &= 2\,t_1\cos.\alpha \\ t_1 &= 2\,t_2\cos.\alpha_2 \\ t_2 &= 2\,t_3\cos.\alpha_3 \\ &\vdots \\ t_{n-1} &= 2\,t_n\cos.\alpha_n = 2\,P\cos.\alpha_n; \end{aligned}$$

hence, multiplying these equations together and expunging the factors common to each side of the resulting equation, we have

$$W=2^n\,P\,(\cos.\alpha.\cos.\alpha_1.\cos.\alpha_2\;.\;.\;.\;.\;\cos.\alpha_n) \;.\;.\;.\;.\; (4).$$

P being the power and n the number of pullies.

If the several cords are parallel, as in fig. 71, this equation becomes $W=2^n\,P$ (5); and, if the angles are all equal to each other, $W=2^n\,P\cos.^n\alpha$ . . . . (6).

In what is here said the weights of the several pullies have been

neglected, but in strictness these should be taken into account. If they increase the weights on the ropes which pass round them by the several quantities A, A_1, A_2, &c., then the above series of equations will be

$$W+A=2t_1 \cos. \alpha_1$$
$$t_1+A_1=2t_2 \cos. \alpha_2$$
$$\&c. \qquad \&c.$$

(85.) Instead of attaching the several ropes to immoveable points, as in fig. 71, they are all in another arrangement of the system fastened to the weight, as in fig. 72, the ropes being parallel. The relation between the power and weight in this system is at once seen from looking at the figure; thus the first pulley or that which first receives the power supports twice P, the second, therefore, supports four times P, the third eight times P, and the nth supports 2^n times P, and all of this except P is the weight; hence deducting P the weight supported is $W=(2^n-1)P\ldots.(7)$.

In this system, as well as in that exhibited in fig. 71, each pulley is connected with two parallel branches of rope, of which each branch bears half the weight attached to the pulley; but if three parallel branches of rope be connected with each pulley, as in the systems exhibited in figures 73 and 74, each branch will bear a third of the attached weight; hence, if n be the number of these systems of threes, we shall have instead of the equations (5) and (7),

$$W=3^n P \text{ and } W=(3^n-1)\ldots.P;$$

the first equation belonging to the arrangement in fig. 73, and the second to that in fig. 74.

(83.) We have hitherto supposed each pulley to be attached to a separate string; but if only one string is employed, as in the systems represented in fig. 75, then, as this string is uniformly tense throughout, the common tension being P, the weight, which is equal to the sum of the tensions when the branches of the rope are parallel, must be equal to 2P times the number of moveable pullies; that is, $W=2nP$, W including the weight of the lower block.

The Inclined Plane.

(87.) This machine is simply a plane surface inclined to the horizon.

The theory of this machine is unfolded in problem I. page 48, where it is shown that if i be the inclination of the plane, and ϵ the angle which the direction of the power makes with the plane, the relation between the power and weight is $P=W\frac{\sin. i}{\cos. \epsilon}$. When

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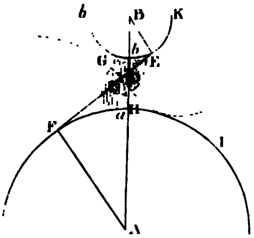

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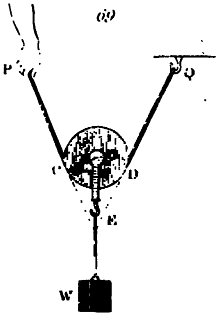

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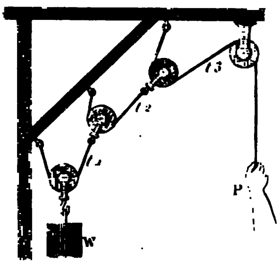

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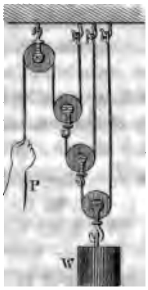

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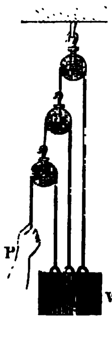

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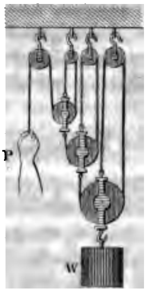

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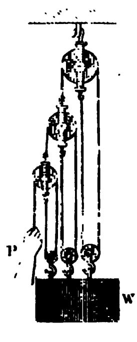

the direction of the power is parallel to the plane, the machine affords the greatest mechanical advantage possible, for then

$$\cos. \iota = 1 \text{ and } P = W \sin. i,$$

so that in this case the power is to the weight as the altitude of the plane to its length.

But when the direction of the power is horizontal or parallel to the base of the plane then $\iota = i$, and the relation is $P = W \tan. i$, which shows that the power is to the weight as the height of the plane to the base.

The Screw.

(88.) Before we can investigate the mechanical power of the screw we must ascertain exactly the form of the spiral surface which it presents. The generation and equation of this surface has been explained in the *Diff. Calc.* p. 200; but it will be proper to repeat, in part, that explanation in this place. Let us conceive then a rectangle to be rolled round a vertical and cylindrical column which it just embraces, the line which was the diagonal of this rectangle will form itself into a winding curve, called a *helix*, and it will make just one turn round the column, its horizontal projection being a circle; if immediately above this another equal rectangle be applied to the cylinder, the vertical edges, when brought together, being in a line with those of the first, the diagonal of this will form a continuation of the helix, and, in this way, will be exhibited on the surface of the cylinder, the trace of the winding surface which forms the screw.

If now, beginning with the bottom, we were to strip off these rectangles one after the other, turning the cylinder round at the same time, so that all of them might be ranged in the same vertical plane, we should, obviously, have the figure presented at fig. 76, the uniform straight line AB being the developement of the helix; we may, therefore, say that this curve is formed by winding round an upright cylinder an inclined straight line AB, always preserving its inclination constant; if we consider this inclined line to be the edge of an inclined plane, then the surface, thus wound round the cylinder, will be the surface of the screw. This surface, therefore, differs from the inclined plane in no respect but in its winding course, and it will, obviously, require just as much power to sustain a weight on the winding surface as on the straight surface, so that if W be any weight on the surface, and a power acting horizontally support it, we must have $\frac{P}{W} = \frac{BC}{AC} = \frac{bc}{Ac}$; but Ac is the circumference of the cylinder which carries the screw, therefore calling the radius of it r, and the height bc, which measures the interval between each turn of the screw and the next, h, we have

$$\frac{\mathrm{P}}{\mathrm{W}}=\frac{h}{2\,\pi\,r}\;\ldots\ldots(1);$$

but if the power P, instead of being applied directly to W, is applied to the arm of a lever, at R distance from the fulcrum, and if the direction of W be at r distance on the other side, then the value of P will be $\mathrm{P}\frac{\mathrm{R}}{r}$, and in this case the condition of equilibrium will be

$$\mathrm{P}\frac{\mathrm{R}}{r}=\mathrm{W}\,\frac{h}{2\,\pi\,r}\;\therefore\;\mathrm{P}=\mathrm{W}\,\frac{h}{2\,\pi\,\mathrm{R}}\;\ldots\ldots(2).$$

Now this equation expresses the power of the screw; for the weight to be balanced or raised, the resistance to be overcome, the pressure to be sustained, &c. is always a force in the direction of the axis of the cylinder, and acting upon the inclined plane winding round it; it is balanced by a power P, acting perpendicular to the same axis, at the extremity of a lever (see fig. 77) whose fulcrum is in the axis, and, therefore, at the distance of the radius of the cylinder from the winding surface or *thread* of the screw. The weight is thrown on the thread by its being connected with a nut or *internal screw* N, which is a spiral-grooved case fitted to receive the external screw.

In fig. 77, the whole pressure on the screw is thrown upon the thread within the nut N, and the power applied at the extremity of the lever R must bear to this pressure or weight the relation (2) above, in order to balance it, or, which is the same thing, a power something greater than this must be applied to move the lever. The relation (2) when expressed in words is this, viz. The power is to the weight or resistance as the interval between two adjacent turns of the thread to the circumference of the circle described by the power; and this relation we see is altogether independent of the radius of the cylinder, and, therefore, also of the degree of protuberance of the thread; it varies only with the distance h between the turns or contiguous spires of the thread, and with the inclination of the thread to the axis of the cylinder; hence so long as this distance and this inclination is preserved, it matters not what form be given to the surface of the thread, nor how protuberant it be made.

We see from the expression (2) that there are two ways in which the power of the screw may be increased; first by diminishing the distance h between the turns, or, secondly, by increasing the length of the lever R; it is, however, not strictly correct to say that the power of the *screw* is increased by this latter change, for this in fact remains unaltered: it is the *applied power* that is here increased.

The Wedge.

(89.) The wedge is a triangular prism, AC (fig. 78), chiefly employed for splitting or separating bodies; for this purpose the edge AB of the wedge is introduced between the bodies to be separated, and the power which drives it is applied to the *head* DC; this power, when the machine is in equilibrium, must balance the resistances opposed to its entry, and which can only act on the edge AB, and on the two *faces* DB, CA. Indeed in a state of equilibrium there can be no resistance opposed to the edge of the instrument, as is obvious, so that this state is preserved by three forces, viz. the power P applied to the head, and the resistances P_1, P_2 acting against the faces. To determine the conditions of equilibrium of forces, thus acting, we must introduce hypotheses, not only unsupported by observation and experiment, but in direct contradiction of them; thus, for three forces to keep a body in equilibrium, it is absolutely necessary that when one of the three is withdrawn, the body, through the influence of the other two, should move unmolested in a direction opposite to that of the force withdrawn; in the wedge, therefore, when the pressure or impelling power is withdrawn from the head the pressures on the sides should expel it from between the resisting surfaces; this, however, is almost universally contrary to experience: in the cleaving of wood, for instance, the wedge may be driven to any extent between the resisting sides (fig. 79), and will usually remain there without sensibly receding, although the power be removed from the head, the *friction* being fully equal to balance the expelling forces acting on the faces.

The mathematical theory of this machine is founded on the hypothesis, that the resisting surfaces as well as the faces of the wedge are perfectly smooth, or, which is the same thing, that the friction is nothing, whereas in practice the friction is every thing, the wedge would be comparatively useless without it. Seeing, therefore, the great effect of friction in this machine, which is sufficient to maintain the equilibrium even when the applied power is withdrawn, it is obvious that no deduction from the mathematical theory of it can be of much practical utility. It is true, indeed, that in all other machines, as well as in the wedge, friction always opposes a hindrance to the full effect of the applied powers to produce motion, and that, therefore, the deductions of pure theory, where these hindrances are not taken into account, require some modification before they can agree with the results of actual experiment, but then, by polishing or lubricating the acting surfaces, these hindrances may be more and more diminished, and the results of practice be made to approach nearer and nearer to the deductions

of theory. In the application of the wedge, however, it is in most cases impossible, even if it were desirable to diminish in the smallest degree the friction on its faces, it is this that hinders the pressures on the faces from driving the wedge back, and is, therefore, a power which greatly favours the efficacy of the machine; as the impelling force is usually applied at intervals, by means of repeated blows, and not in the form of a continued pressure, the friction serves to hold the wedge where the last blow had driven it.

(90.) Abstracting from the influence of friction, the equilibrium of the wedge may be thus investigated. Let any arbitrary length, DE, (fig. 80,) represent the power applied perpendicularly to the head of the wedge, and draw DM, DN perpendicular to the faces AC, BC, and complete the parallelogram IK, having DE for its diagonal; DI and DK, or IE, will then represent the pressures P_1, P_2 against the faces of the wedge, so that the three equilibrating powers are as the three sides of the triangle DEI, or as the three sides of the similar triangle ABC, that is,

$$P : P_1 : P_2 :: AB : AC : BC;$$

or, calling the length of the edge C, l,

$$P : P_1 : P_2 :: AB \times l : AC \times l : BC \times l;$$

which proportion, obviously, implies that the power on the head of the wedge and the equilibrating pressures on the faces are proportional to the areas of the head and faces, on which they respectively act.

SCHOLIUM.

To the foregoing theory of the simple machines we shall append the following judicious remarks from *Venturoli*.

The false opinion which persons unskilled in the nature and the power of machines are apt to conceive, often encourages empty errors and mischievous deceptions. One of the most common of these conceits is that of considering machines as available to increase and multiply the force of agents, which is not always true. To form a just notion of the aid which may be expected from machines, looking to the uses to which they are most commonly put, we shall divide them into two classes; those intended simply to sustain a weight, and those intended to draw it, or raise it equably.

In machines of the first class, both the effect of the machine and the immediate effect of the power can only be estimated by the weight sustained. This being understood it is evident that the machine increases the effect of the power; so that, for example, a force of 10 lbs. will sustain by means of a lever, 100 lbs., provided that the arm of the force be ten times as long as that of the weight.

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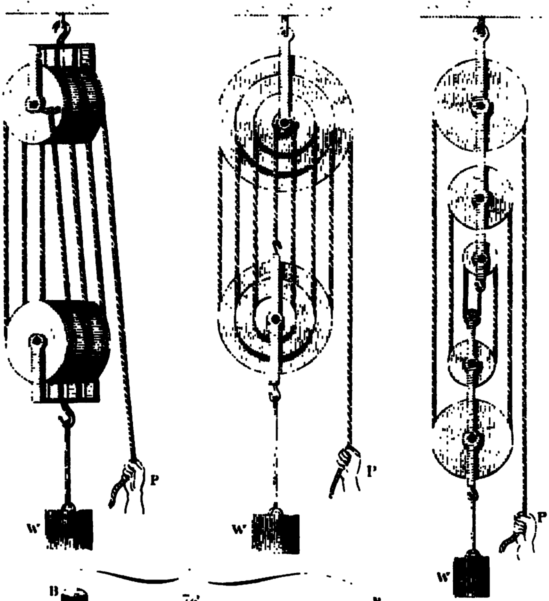

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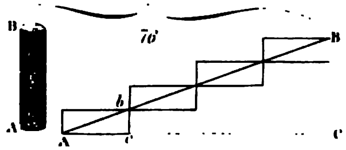

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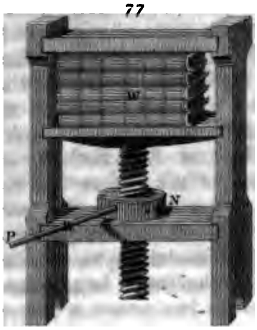

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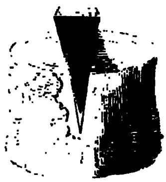

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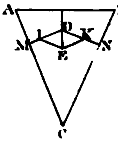

If it be asked how the force can ever produce an effect so much greater than itself, we shall perceive, if we consider well, that the force 10 does not really sustain the whole weight 100, but only the tenth part of it. Let the lever be supposed to be of the second kind; the force 100 may be resolved into two, the one equal to 90 which acts upon the fulcrum, and the other equal to 10 which acts at the point of application of the power. The first is entirely sustained by the prop, and the power sustains the second alone. Archimedes required only a fixed point to hold the terraqueous globe in equilibrium. If he had found it, says Carnot, it would not in reality have been Archimedes, but the fixed point, which would have sustained the earth.

In machines of the second class neither the effects of the machine nor that of the power can be estimated simply by the weight raised; otherwise the measure of the effect would be altogether vague and indeterminate. In fact any force, however small, may carry a weight of any assignable magnitude however great; if it only be granted that the weight admits of being divided and of being carried, one piece at a time. Wherefore it is necessary to take into account the time also in which the power can carry the weight through a given space, or the velocity with which the weight is carried; and on this account it is that the effect is measured by the product of the weight multiplied by the velocity.

Now upon this principle we have already shown that the machine does not increase the effect of the force. If a man with a force equivalent to 10, raise, by means of a machine, a weight of 100, he moves with a velocity ten times as great as that of the weight, and does as much as if operating without any machine he carried those 100 at ten journeys, loading himself with 10 at a time. In a word, what is gained in the quantity of the weight moved is lost in the velocity; and the effect remains the same.

Between the two classes of machines, above described, there is then this characteristic difference, that the first add to the effect of the power, the second do not add to it.

There is another difference, not less remarkable, respecting the resistances of friction, and of ropes, and other resistances. In machines of the first class these resistances are all of them advantageous to the power* and themselves also sustain their portion of the weight; whence there remains so much the less of it for the power to support. On the contrary, in machines of the second class, the resistances are all of them detrimental to the power, and form part of the weight to be overcome: whence, on this account, a force is re-

* Because the weight, before it can move, must overcome these resistances as well as the power.

quired greater than that which would be required in the immediate application of the power.—*Venturoli's Mechanics*, part ii. p. 164.

CHAPTER V.

ON THE STRENGTH AND STRESS OF BEAMS.

(91.) It is obvious that in all the practical operations of mechanics, and more especially in the raising of structures, it is of great importance to know the weight or stress each component part is fitted to bear without endangering the stability of the whole; and, consequently, numerous experiments to ascertain the strength of materials, particularly of beams and bars, have at various times been undertaken by scientific men. Into a detail of these experiments we do not, however, propose to enter, but merely to present to the student, in a short compass, some of the more interesting and valuable particulars furnished by theory, and confirmed by the experiments adverted to.

If a uniform rod or bar of any substance be suspended by one extremity, and loaded at the other till it is on the point of being torn asunder, we ought to expect, independently of actual experiment, that the weight would be proportional to the tranverse section; for, if this bar were conceived to be divided longitudinally into any number of equal strips, no reason could be assigned why one of these should support a greater portion of the weight than either of the others, so that each would support an equal part of the weight supported by the whole, just as an assemblage of parallel ropes divide the weight of an appended body equally among them. All experiments on lateral strains prove this deduction to be correct, and to be quite independent of the figure of the section, requiring only uniformity and equality in the texture of the bodies compared, so that we may lay it down, as a general law, that in bars of the same material the lateral resistances are as the areas of their traverse sections.

(92.) When the bar or beam is supported in a horizontal position, then the law of resistance, which it opposes to fracture by an incumbent weight, is more difficult to establish, because here we do not see so clearly how the resisting forces exert themselves, nor in what degree. It was laid down by *Galileo* that if a beam were supported at its extremities, as in fig. 81, and loaded by a weight at the middle, that all the fibres of the beam would exert equal resistances to prevent fracture, and that when these were overcome the section

would tend to turn about that boundary of it in contact with the weight, viz. about AB. As all the fibres exert equal resistances, and in the direction of their lengths these resistances will be so many equal and parallel forces which may, therefore, be considered as concentrated in the centre of gravity of the section, so that denoting the resistance of a single fibre by k, and considering the section to be a rectangle of breadth b, and depth h, kbh will express the sum of the resisting forces, and as this acts at the centre of gravity its moment to turn the section about AB will be

$$kbh \times \tfrac{1}{2} h = \frac{kbh^2}{2}.$$

(93.) The hypothesis of *Leibnitz* agreed with that of Galileo, as regards the axis about which the section would turn, but it differed from it as regards the equal resistances of the fibres throughout the whole fracture; for, according to Leibnitz, the forces exerted by the fibres were directly proportional to their distances from the axis of the section, so that the middle fibre exerted but half the force of the extreme fibre, therefore, calling the force of this k, the sum of the forces would be $\frac{kbh}{2}$, and the centre of such a system of parallel forces being at $\frac{2}{3}h$, the moment to turn the section would be $\frac{kbh^2}{3}$.

Now it may be remarked that as far as regards the comparative strength of rectangular beams of the same material, or of beams generally, which have only rectangular sections when cut traversely it matters not which of these hypotheses be adopted, for both equally warrant the inference that *the law of resistance is as the breadth, multiplied by the square of the height or depth;* and this law, which has been confirmed by numerous experiments, immediately leads to an inference of considerable practical importance, viz. that a beam is much more efficient when placed with its narrower side uppermost, that is, so that its breadth may be less than its depth; for if we call the breadth b, and the depth or height h, then the relative strengths, when b and h are alternately uppermost, are expressed by bh^2 and hb^2, and, consequently, in the former position the beam is $\frac{h}{b}$ times as strong as in the latter, that is, as many times as strong as the depth contains the breadth. This important fact is always attended to in buildings, the joist, rafters, &c. being always placed with the narrower side uppermost.

It has been supposed above that the segments of a fractured beam tend to turn about the line where the fracture terminates; but, from

experiments recently undertaken by Mr. Barlow, it appears that AB (fig. 82) is not the line about which the section tends to turn, as Galileo and Leibnitz had supposed, but that the tendency is to turn about a line entirely within the section, so that the fibres on that side of the line where the fracture begins are extended, and those on the other side compressed; this axis Mr. Barlow calls the *neutral axis*, dividing the section into the *area of tension* and the *area of compression*, and he calls the *centre* of tension or of compression that point in the area of tension or of compression where all the forces in that area should be collected to have the same effect, or the same moment, with respect to the neutral axis.

The existence of a neutral axis *somewhere* within the area of fracture, was maintained by *Mariotte*, *James Bernoulli*, and Professor *Robison;* but Mr. Barlow appears to have been the first who set about the determination of this axis by actual experiment. (See the historical sketch of former theories, prefixed to Mr. Barlow's Essay the Strength and Stress of Timber.)

The general conclusion from these experiments was this, viz. "The centre of tension and the centre of compression, each coincided with the centre of gravity of its respective area: and the neutral line, which divides the two, is so situated that the area of tension into the distance of its centre of gravity from the neutral axis is to the area of compression into the distance of its centre of gravity from the same line, in a constant ratio for each distinct species of wood, but approximating in all towards the ratio of three to one.

(94.) This theorem being established, says Mr. Barlow, it is evident that we may thence, without any specific numbers for exhibiting the actual resistance of the fibres, compute the proportional strengths of differently formed beams; and of the same formed beams in different positions; of which we will give one example by way of illustration.

Problem I.—Let a square beam be fixed with one end in a wall, first in a direct position, viz. with its sides perpendicular and horizontal; and, secondly, with its diagonal vertical to find the ratio of its strength in these two positions.

Conceive ABCD (fig. 83) to denote the beam in its first position, EF the neutral axis, EABF the area of tension, and *t* the centre of tension or centre of gravity of that area, EDCF the area of compression, *c* its centre of gravity, and G the centre of gravity of the whole area of fracture, and the same letters will denote the similar quantities, in fig. 84, which represents the section of the beam in its second position.

Then, by the preceding theorem, we have (fig. 83),

$$\textit{area}\ \mathrm{AEBF}\times nt\times 3=\textit{area}\ \mathrm{EDCF}\times nc,$$

and in fig. 84, $\textit{area}\ \mathrm{EBF}\times nt\times 3=\textit{area}\ \mathrm{EFCDA}\times nc$;
both which, from the property of the centre of gravity at (p. 59), are reducible to

$$\left.\begin{array}{l}(\text{fig. } 83,)\\(\text{fig. } 84,)\end{array}\right.\begin{array}{l}\textit{area}\ \mathrm{AEBF}\times 2\,nt=\textit{area}\ \mathrm{ABCD}\times n\,\mathrm{G}\\ \textit{area}\ \mathrm{EBF}\ \ \times 2\,nt=\textit{area}\ \mathrm{ABCD}\times n\,\mathrm{G}.\end{array}$$

For the sake of simplifying the computation, let the side of the square $=1$, $n\mathrm{H}=x$, or $nt=\frac{1}{2}x$, then $n\mathrm{G}=\frac{1}{2}-x$; the area AEBF $=x$, and the area ABCD$=1$, whence our first equation gives $x^2=\frac{1}{2}-x$, or $x^2+x=\frac{1}{2}$; whence $x=-\frac{1}{2}\pm\sqrt{3}=\cdot366$, which denotes both the depth of tension Hn, and area of the same AEFB; consequently, $\cdot366\times\frac{\cdot366}{2}=\cdot066978$, the numerical expression for the resistance to tension, on which depends the strength of the beam. It remains now to compute the same for the second position of the beam, as in fig. 84.

Here if we denote nB by x, $nt=\frac{1}{3}x$, and the area EBF$=x^2$, also area ABCD$=1$, as before; whence our second equation becomes

$$x^2+\tfrac{2}{3}\,x=\tfrac{1}{2}\sqrt{2}-x,\ \text{or}\ x^2+\tfrac{3}{2}\,x=\tfrac{3}{4}\sqrt{2}=1\cdot0606.$$

From which we readily obtain $x=\cdot578$ nearly,
$x^2=\cdot33408$, and $x^2+\frac{1}{3}\,x=\frac{1}{3}\,x^3=\cdot06436$, which is the numerical value of the tension in this position of the beam. The strength of the beam, therefore, in the latter position is to that in the former in the ratio of $\cdot06436:\cdot06697$; or as the numbers $643:669$ *nearly*, which accords with experimental results, and, in a similar way, may the strengths of differently formed beams be compared. We shall now consider the straining effects on beams differently supported, and loaded by weights at different parts.

Problem II.—A beam of timber AB (fig. 85) is fixed with one end in a wall, and loaded with a weight W at the other end; to determine the efficacy of this weight to break the beam.

At the place where the beam has the greatest tendency to break, the broken piece will tend to turn round the neutral axis; if the distance of this axis from W be l, then lW will express the energy of W to produce this effect; but lW will be greatest when l is greatest, that is when this denotes the whole length of the projecting beam; hence the beam will tend to break close to the wall, and lW will express the strain there. The strain varies therefore as the length of the beam, W being the same.

Problem III.—A beam rests loosely on two props, A, B (fig. 86), and is loaded at a given point C by a given weight W: to determine the stress at C.

Of course the tendency to break will be at C, and we may, therefore, assimilate this to the preceding case by conceiving the beam to be fixed in a wall up to C, and to be strained by a force equal to the pressure upon the prop at the other end.

Now by (36) the pressure on the prop A is expressed by

$$P=\frac{CB}{AB}W:$$

this then being the force that strains the projecting beam AC, its energy is $\frac{AC\,.\,CB}{AB}W$.

If the load be in the middle of the beam, the product AC . CB becomes the greatest possible, being the square of half the length; hence a beam will be less able to support a weight at its middle than if it be placed at any other part. The strain obviously varies as the product of the distances of the weight from the props.

It may be further remarked, that when the weight acts at the middle, the stress, being $=\frac{AC^2}{2\,AC}W=\frac{1}{4}\,AB\,.\,W$, is one-fourth the stress which the same weight W would produce acting at the extremity of a projecting beam equal to AC, or it is equal to the whole stress on a projecting beam equal to AB, the weight at its extremity being $=\frac{1}{4}$ W, or a projecting beam equal to half AB and loaded with $\frac{1}{4}$ W at its end, will suffer half the stress at the wall.

PROBLEM IV.—To determine the stress on a projecting beam, and on a beam resting on props when the weights are distributed uniformly over them.

Let AC be the projecting beam and w its weight, including the uniform load; this weight will act at half the distance AC from the wall, and therefore the stress is $\frac{1}{2}$ AC . w, which is just half what it would be if w were placed at the extremity.

When the beam rests on props (fig. 87), the pressure on each prop is half its weight W, this, therefore, is the force acting at P which tends to fracture the beam at C with an energy expressed by

$$\tfrac{1}{2}\,W\,.\,AC\,;$$

but there is another force exerted, viz. that due to the weight w of the portion CA, and which, by last case, opposes the former with an energy expressed by $\frac{1}{2}\,w\,.\,AC$; hence the expression for the stress on C must be $\frac{1}{2}\,(W-w)\,AC$: to eliminate w we have, on account of the uniformity of the beam,

$$AB : AC :: W : w=\frac{AC}{AB}W,$$

so that the expression for the stress will be

$$\frac{1}{4} W \left\{ \frac{AB \cdot AC - AC^2}{AB} \right\} = \frac{1}{4} W \frac{AC \cdot BC}{AB}.$$

Hence in this case also, as in problem III., the stress varies as the product AC . BC.

The strain at the middle point of the beam is $\frac{1}{8}$ W . AB, just half what it would be if the whole load were placed there, (problem III.)

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(95.) By means of these problems it will be easy to find the most economical forms for beams, either projecting or supported at the ends, so that they may in no part possess superfluous strength, that is, that the strength in every part may be exactly in proportion to the stress there. Thus if a projecting beam of uniform breadth is to support a weight at its extremity, it will be equally strong throughout if the vertical sides are in the form of a parabola (fig. 88); for, by (prob. II.), the stress varies as $AC = x$, and the strength of the beam, being (93) as the breadth into the square of the depth, and the breadth being constant, varies simply as $CD^2 = y^2$; hence, in order that the strength and stress may be throughout in a constant ratio, AC must vary as CD^2, that is, $\frac{CD^2}{AC} = \text{constant} = a$ or $y^2 = ax$, the equation of a parabola. But the shape need not necessarily be parabolic in order to insure uniformity of strength, as it will depend in a measure upon the nature of the vertical sections: thus, if these sections are required to be all squares, then the breadth and depth being every where the same, the strength will vary as the cube of the depth, and hence AC should vary as CD^3, that is, $y^3 = ax$, the equation of the cubical parabola, which must therefore be the form of the tapering beam.

Again, if the depth of the beam is to be constant, then AC should vary as the breadth, and therefore the upper and under faces of the beam will be triangles. If a beam supported on two props is to be uniformly strong, and at the same time uniformly broad, it will be necessary to form the vertical sides elliptical, for the breadth being constant the strength will vary as the square of the depth, and, by (prob. IV.) the stress at any point C varies as AC . CB; hence the square of the depth at C must be in a constant ratio to AC . CB, which requires that D (fig. 89,) be always in an ellipse whose axis is AB, (*Anal. Geom.* p. 122.)

(96.) It may be moreover remarked, that, by means of the foregoing expressions for the strain or tendency to produce fracture, combined with the results of experiment, we may determine the

actual weight which any given beam will support in given circumstances. Thus, suppose it is found by experiment that a beam of breadth b, depth h, and length l, just breaks with a weight w at its middle, and that it is required to determine what weight W will just break another beam of like materials whose breadth is B, depth H, and length L. In each case the tendency to *resist* fracture is just balanced by the tendency to *produce* it; the expressions, therefore, for these two tendencies must be equal, and therefore the ratio of the tendencies to resist fracture in the two beams must equal the ratio of the tendencies to produce fracture. Now the tendency to resist fracture is what we understand by the strength of the beam, and the tendency to produce fracture is the stress; hence, equalizing the two ratios spoken of, we have (prob. III.)

$$\frac{B \,.\, H^2}{b \,.\, h^2}=\frac{\frac{1}{4} L \,.\, W}{\frac{1}{4} l \,.\, w} \;\therefore\; W=\frac{B \,.\, H^2}{b \,.\, h^2}\cdot\frac{l \,.\, w}{L},$$

the weight required. It is obvious that from the same equation we may deduce the length L when W is given, and also that the equation remains the same whether W, w act in the middle or at the end of each beam.

The expression here given may serve to compare the strength of any beam in a *model* with that of the corresponding beam in the structure. Thus, suppose the beam which we have considered to have been submitted to experiment to belong to the model, and the other to be the corresponding beam in the structure whose like dimensions are n times those of the former, then the foregoing expression for W will be $W=n^2 w$, which will be the greatest possible load the beam in the structure can bear, including of course its own weight. Now if the weights alone of the two beams are respectively p and P, then, since these must be as the cubes of their like dimensions, we must have $P=n^3 p$, consequently the beam in the structure so far from bearing a load, will but just support its own weight if we make it so large that $n^3 p=n^2 w$, that is, if $n=\frac{w}{p}$: we see, therefore, how erroneous it would be to estimate the strength of a large beam in a structure by that of a similar small beam in a model, regarding only the comparative dimensions of each; for, by increasing the magnitude of the large beam, without in the least changing the relative proportions of the two, we should nevertheless render it at length too large to support even its own weight, although the model agreeably to which it has been formed, might be able to support a load many times its own weight. There is, therefore, necessarily a limit to the magnitude of all structures, even indeed to the magnitude of the animal structure, and to trees, beyond which limit they would be unable to support their own

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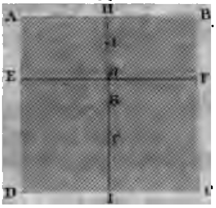

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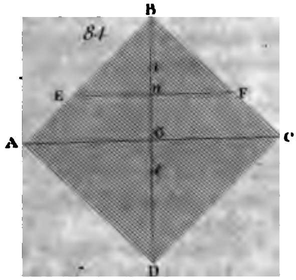

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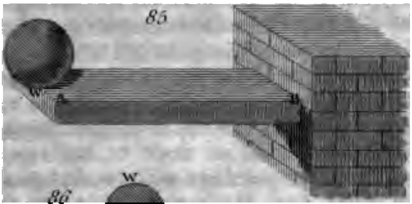

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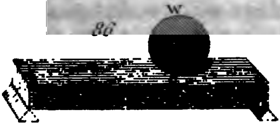

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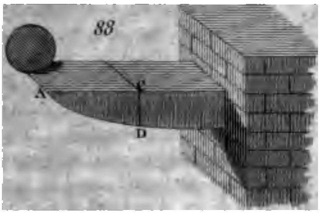

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weight; we accordingly find men of enormous magnitude, as O'Brien "the celebrated Irish Giant," to be so weak that they are scarcely able to walk about.

In connexion with these remarks may be mentioned the curious question, proposed by Mr. Emerson, among the mechanical problems annexed to his algebra; the question is this: Supposing, with *Borelli*, that a strong man can bear but 26lbs. at arm's end, and that the weight of his whole arm is equivalent to 4lbs. at arm's end; from the length of his arm being given, to find the dimensions of that man's arm that can bear no more than its own weight.

This problem is immediately solved by means of the relation $n = \frac{w}{p}$, deduced above, w representing here the weight 26+4 or 30lbs., the weight of the common man's arm and load, and p representing the weight 4lbs. of his arm alone, so that $n = 7\frac{1}{2}$: this, therefore, is the number of times any dimension of the large man's arm must contain the corresponding dimension of the common man's arm; let us then suppose the common man's arm to be a yard long, the length of the other man's arm, to just support itself, must be $7\frac{1}{2}$ yards, and, as the body is, in well proportioned persons, about twice as long as the arm, we therefore conclude that a man upwards of 15 yards high would not be able to stretch out his arm.

Problem IV.—To determine the relative strengths of beams loaded in the middle when their ends are loosely supported, and when they are firmly fixed in two vertical walls.

When a beam is loosely supported and acted upon by a weight at the middle, this *weight* is equally divided between the two props, but at these points there is no *strain*; when, on the contrary, the ends of the beam are firmly fixed in immovable walls, then it is the *strain* on the middle which is equally divided between the two extremities; that is to say, the fibres in the section at each wall are strained half as much as those at the middle section. The whole of the weight, therefore, is not expended here, as in the former case, in straining the middle of the beam, but a portion is employed in straining each end half as much. Now, whatever weight strains the middle, $\frac{1}{4}$ of this will (by prob. III.) strain each section at the wall half as much; hence, if we represent that part of the weight which strains the middle only, by 4, the part which strains the ends will be 2, and therefore the whole straining weight will be 6, so that the weight 6 will produce no more stress on the middle of the beam thus fixed, than the weight 4 when the ends rest loosely on props; hence the relative strengths of fixed and loose beams are as 6 to 4 or as 3 to 2, which relation *Mr. Barlow* has verified by experiment.

It may be observed, that the weight 8 uniformly distributed over the beam would produce the same strain in the middle as the weight 4 applied there (prob. IV), and the weight 4 uniformly distributed will strain the ends as much as the weight 2 applied to the middle: hence, if the load be uniformly distributed over the fixed beam, it will be no more strained with the weight 12 thus disposed, than with the weight 6 acting at the middle, so that here, as in the other cases, the efficiency of the beam is doubled by spreading the weight uniformly over it.

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(97.) The result of the preceding investigation, although confirmed by Mr. Barlow's experiments, differ materially from the conclusions deduced by other philosophers, as *Girard*, *Emerson*, and *Robison*, who find the comparative strengths of supported and fixed beams to be as 1 to 2, and not as 2 to 3. Emerson's reasoning on this point is as follows:

Suppose DA=AC (fig. 90,) and BE=BC, and let P be the weight which would break the beam when resting on A and B. Suppose the beam cut through at C, and let $\frac{1}{2}$ P be laid upon D, whilst $\frac{1}{2}$ P remains at C; then the pressure at A will be =P, therefore the beam will also break at A having the same stress there as it had at C. For the same reason, if $\frac{1}{2}$ P be applied to E, CE will break at B. Consequently, if 2 P be applied to C, the beam being whole, and the ends D, E fixed, the beam will break at A, C, and B; and, therefore, bears twice the weight, or 2 P at C, before it breaks.

Now the foregoing reasoning appears to assume, that before the beam can break at C, the strain on A and on B must be sufficient to break the beam at those points also; yet it is shown that the beam will break simultaneously at these three points, if, besides the weight P acting at C, the points D, C, E, be each loaded with $\frac{1}{2}$P; hence, to enable the middle point C to yield to the pressure of P, it is only necessary that the fibres at A and at B be half as much strained as they are by the influence of $\frac{1}{2}$ P acting at D, C, and E; because but half the strain of the fibres at A are, in virtue of this influence, in the direction of AC, the other half being in the direction of AD; and, in like manner, but half the strain at B is in the direction of BC, so that if, in addition to P acting at C, as much more weight is added as will produce these half strains, the parts AC, BC will be deflected sufficiently for the beam to break at C; we have, therefore, to add to P only half as much as would produce the whole strains at A and B, that is, instead of P we should add $\frac{1}{2}$P, making the whole breaking load $\frac{3}{2}$ P, which is the same result

as before obtained, and the correctness of which Mr. Barlow's experiments confirm.

Mr. Barlow remarks on this subject, "in every experiment that I made after the complete fracture in the middle, the two fragments had been so little strained at the points of fixing, that they soon after recovered their correct rectilinear form;" and, in order to show the foundation of the error which all theorists have made, in assuming that the fixed beam would break simultaneously in the middle and at the walls, he further adds, "If the beam instead of being fixed at each end were merely rested on two props, and extended beyond them on each side equal to half their distance, and if weights w, w' (fig. 91,) were suspended from these latter points each equal to one fourth the weight W, then this would be double of that which would be necessary to produce the fracture in the common case; for, dividing the weight W into four equal parts, we may conceive two of these parts employed in producing the strain or fracture at E, and one of each of the other parts as acting in opposition to w and w', and by these means tending to produce fractures at F and F'.'

"This is the case which has been erroneously confounded with the former, but the distinction between them is sufficiently obvious; because, here the tension of the fibres, in the places where the strains are excited, are all equal; whereas in the former the middle one was double of each of the other two."*

Venturoli, in his valuable book on Mechanics, says, in the words of Dr. Creswell's translation, "The beam would sustain a load considerably greater, if, instead of being simply placed upon two props, it were immoveably fixed in stone-work at both its extremities. For, it that case, it cannot break unless it gives way in three places at the same time."†

Problem V.—To determine the dimensions of the strongest rectangular beam that can be cut out of a given cylindrical tree.

Let r be the radius of the base of the cylinder, and x and y the breadth and depth of the required beam, then, as the strength varies as xy^2, this quantity must be a maximum; hence

$$y^2+2\,xy\frac{dy}{dx}=0 \therefore y+2x\frac{dy}{dx}=0.$$

Also, as the diagonal of the rectangle is equal to the diameter of the circle, we have

$$x^2+y^2=4r^2 \therefore x+y\frac{dy}{dx}=0 \therefore \frac{dy}{dx}=-\frac{x}{y};$$

* Essay on the Strength and Stress of Timber, third ed. p. 149.

† *Venturoli's Mechanics*, Part II., p. 60.

hence, by substitution, $y - \frac{2x^2}{y} = 0 \therefore y^2 = 2x^2 = 4r^2 - x^2$

$\therefore 3x^2 = 4r^2 \therefore x = \frac{2r}{\sqrt{3}}$, $y = 2r\sqrt{\frac{2}{3}}$, the dimensions required.

For further information on the subject of this chapter, and more especially for an account of the various experiments that have hitherto been made to determine the strength of materials, the student is referred to Professor Gregory's valuable Treatise of Mechanics; to the second volume of Sir David Brewster's edition of Ferguson's Lectures; to Mr. Barlow's work on the Strength and Stress of Timber, as also to his treatise on Mechanics in the Encyclopædia Metropolitana; to Part II. of Creswell's translation of Venturoli; and lastly, to Professor Leslie's instructive volume on the Elements of Natural Philosophy.

Perhaps we ought to remark before closing this chapter, that in all the foregoing investigations on the stress of beams, we have not taken into account the deflection from the horizontal line which the force produces before it actually breaks the beam. By reason of this deflection the energy of the breaking force is not, strictly speaking, expressed by the intensity of the force multiplied by its distance measured along the beam from the section of fracture, but by the intensity into the *perpendicular* distance of the fracture from its direction; this perpendicular distance is equal to the former distance multiplied by the cosine of the angle of deflection, and therefore, by introducing this cosine as a factor into all the foregoing expressions into which the moments of the straining forces enter, they will become rigorously correct; but except in very long beams, or in very elastic ones, the deflection is too small to render this modification of much consequence: Mr. Barlow, however, has not neglected its influence in his important inquiries on this subject.

END OF THE ELEMENTS OF STATICS.

PART II.

ELEMENTS OF DYNAMICS.

SECTION I.

ON THE RECTILINEAR MOTION OF A FREE POINT.

(98.) HAVING considered the general theory of equilibrating forces, we come now to DYNAMICS, the second principal division of the science of Mechanics, and which comprehends the theory of unbalanced forces. Dynamics, therefore, considers bodies in a state of motion, while Statics has to do only with bodies at rest; in this first section we shall confine ourselves to the consideration of rectilinear motion only, but in the opening chapter we shall lay down a few general and fundamental principles which always hold, whatever be the path of the moving point, and which, in fact, will be found to comprise the whole theory of its motion.

CHAPTER I.

ON THE FUNDAMENTAL EQUATIONS OF MOTION.

(99.) BY the *inertia* of matter is meant its incapability of altering the state into which it is put by any external cause, whether that state be rest or motion.

It is manifest that if a body at rest receive an impulse in any direction,* it will, if entirely at liberty to obey that impulse, move in that direction, and with a uniform rate of motion; for as we suppose the body to be entirely uninfluenced by any other cause, and since it is incapable of exertion itself, it is plain that for whatever reason we could suppose the motion to slacken at any point of its path, for the same reason we might suppose the motion to quicken. The body will, therefore, continually move at a uniform rate in the direction impressed upon it, that is, if nothing extraneous interferes with its motion.

* The body is here considered as a single point, or else as receiving its impulse towards the centre of gravity, so that no *rotation* is impressed on it.

(100.) We have just spoken of the *rate* of a body's motion: we estimate this, when the motion is uniform, by the space the body passes over in some determinate portion of time, as in one second, which indeed is the portion generally assumed for the unit of time; so that when we observe a moving body to pass uniformly over ten feet every second of time, we express the rate of its motion by saying that it moves with a *velocity* of ten feet, or, for greater brevity, that its velocity is ten feet, and this is what we are to understand by the equation $v=10$ feet, space being taken as the measure, or representative, of velocity.

Suppose now that t represents, not the time, but an abstract number expressing the *number of seconds* elapsed since the commencement of the uniform motion, and let s denote the corresponding space passed over by the body, then we obviously have the three equations

$$v=\frac{s}{t},\ s=tv,\ t=\frac{s}{v},$$

so that any two of the three quantities s, t, v, being known, we may immediately find the third.

But if t'' is not reckoned from the commencement of motion, but only after a certain space s' has been described, then, s being the whole space gone over from the commencement, the three equations will be

$$v=\frac{s-s'}{t},\ s=s'+vt,\ t=\frac{s-s'}{v}.$$

These equations, or indeed any one of them, comprehend the whole theory of the motion of a body acted on by a single impulse, or influenced by any cause which produces uniform motion. We shall give an instance of their application.

Two bodies a, b (fig. 92), animated by the velocities v, v' set out simultaneously from the points A, B, and move in the same direction AC; to determine the time of their coming together.

Suppose they come together at the point C, then

$$AC=vt,\ BC=v'\,t, \text{ that is, calling AC, } s, \text{ and AB, } s'$$

$$s=vt,\ s-s'=v'\,t\ \therefore vt-s'=v'\,t\ \therefore t=\frac{s'}{v-v'},$$

that is, the abstract number expressing the units of time will be that which arises from dividing the space between the points of starting by the difference of the spaces denoting the velocities.

It may be remarked here, that whatever be the nature of the influence which produces uniform motion, and which we have above called an impulse, we have a right to conclude that its effect will be proportional to its intensity; in other words, that such influences, acting on the same body, or on equal bodies, are proportional to the velocities they produce

For if a body receive a certain velocity in consequence of a certain impulse, it ought obviously to acquire double that velocity if at any point of its path that impulse be repeated in the same direction, but if this second impulse take place at that point from which the body set out, it must unite with the first impulse, so that the consequence of a double intensity of impulse will be a double velocity in the body, and, in like manner, a triple intensity will produce a triple velocity, and so on.

(101.) Let us now consider the circumstances of variable motion, and let us first ascertain the expression for the velocity of a body so moving at any epoch t''. If we first assume that the velocity which the body has at t'' continues uniform from t'' to t''_1, then, calling $t_1 - t$, Δt, and the increment of the space or $s^1 - s$, Δs, we have for the velocity at t''; $v = \frac{\Delta s}{\Delta t}$, however small Δt, and consequently Δs, which depends on it, may be; but if no interval of time $\Delta t''$ exists so small, during which the velocity does not vary, then the above equation is true only when Δt, and consequently Δs, becomes 0; hence, by the principles of the differential calculus, we have in this case $v = \frac{ds}{dt}$. (1), which is therefore a general expression for the velocity of a moving body at any time t'' however its motion may vary, and of course it applies also when the motion is uniform, for then

$$v = \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \text{constant} \ldots . (2).$$

(102.) It is obvious that if the velocity of a moving body continually vary, it must be influenced by some continuous cause, however this cause may itself vary in efficiency; for from the instant the cause ceases to act, that instant the body ceases to vary in velocity in consequence of its inertia. We call the cause of variable motion, whatever it really be, *force:* an *accelerative force* if the velocity continually increase, and a *retardive force* if the velocity diminish. We shall, in our general reasonings, consider the force as accelerative, because in order to adapt our conclusions to retardive forces, it will be necessary merely to prefix to the expression for F the negative sign. Let us now investigate this expression; and first we must remark, that as the effect of a constant accelerative force is obviously to generate constant increments of velocity in equal times, if we agree as heretofore to represent causes by their effects, we shall obtain the expression for F by dividing the increment of the velocity by the units in the increment of the time, measured from any epoch t'', that is,

$$F=\frac{\Delta v}{\Delta t}\ldots.(1):$$

such then is the expression for a constant accelerative force, $\Delta t''$ being any interval of time from t'', and Δv the corresponding augmentation of velocity. The velocity of the body in this case is with propriety called a *uniformly accelerated velocity.*

(103.) But suppose that F is not a constant force; then if from any epoch t'' there is an interval $\Delta t''$ so small that F remains unchanged throughout it, the expression just given will in that case represent the intensity of the force acting at the epoch t'', and continuing unabated and unaugmented during the interval $\Delta t''$. If, however, choose $\Delta t''$ as small as we will, F still changes during the interval, then we shall express this fact by saying that the interval $\Delta t''$, during which F remains constant, is 0; hence, for a continually varying force, the expression is $F=\frac{dv}{dt}$ (2), and this may be regarded as a general expression for the accelerative force whether it be constant or variable, for when it is constant

$$F=\frac{\Delta v}{\Delta t}=\frac{dv}{dt}=\text{constant}\ldots.(3).$$

We may give a different form to the general expression for F, for since

$$v=\frac{ds}{dt}\ \therefore\ F=\frac{d^2s}{dt^2}\ldots.(4).$$

We have seen (equa. 1,) that the expression for F at any epoch t'' is equal to the increment of the velocity that *would be* generated in any number of seconds after that epoch (if F were thence to cease to vary,) divided by that number; that is, F, estimated at any epoch t'', is equal to the increment of velocity that would be generated by that force constantly acting during one second. But the velocity of a moving body at any epoch is measured by the space it would pass over in the succeeding second, if its motion were thence to become uniform; hence the force acting upon a moving body at any epoch t'', is measured by the space the body would pass over in the 2d second of time after t'', provided it were to proceed during that second with the increment of the velocity generated during the 1st second.

It thus appears that both velocity and force may be measured by space, and therefore that in every dynamical inquiry, where the mass is not considered, the only concrete quantity concerned is space, for, as before observed, t denotes an abstract number, viz. the number of units or seconds in the time t''.

(104.) It should be remarked here, that the forces of which we have just spoken are in no respect influences of a different kind from those considered in statics; they merely manifest themselves

differently by producing different effects, and it is to the effects only that we look in estimating these influences. The *statical* effect of a force applied to a body is pressure or weight, and we accordingly represent the force, in statics, by pressure or weight. The *dynamical* effect of the same force is accelerated velocity, and accordingly we represent the force by velocity; or, since space measures velocity, we represent it by space. These different modes of estimating the same force, therefore, naturally present themselves upon observing their effects; but, for all the purposes of comparison, it matters not, as was observed in Statics (4), by what we represent the efficiency of any force, taking care only always to keep up the proportion between the forces and their representative quantities. Thus there would be no impropriety, if there were no inconvenience, in representing an accelerative force by a weight, provided we always proportioned the weight to the efficiency of the force; and this leads us to a remark of some importance, viz. that *the pressure or weight produced by the action of a force on any body, is to the pressure or weight produced by the action of any other force on the same body, as the acceleration produced by the former force is to the acceleration produced by the latter:* for it is plain that the ratio of the two forces must be the same abstract number however they are represented; so that if we know the two pressures or the two weights which the forces are fitted to produce, and also the acceleration which one is fitted to produce, we know also the acceleration which the other is fitted to produce.

(105.) In all the foregoing investigations it should be remarked, that we have put entirely out of consideration the nature of the path which the moving body describes. All that we have said as to the velocity of a body regards its rate of motion along the path, whether straight or curved, in which it happens to move, and has nothing to do with the manner in which that motion has been produced; for however it moves, and by whatever agency, the same velocity is always expressed by the same linear space. So too with regard to the moving influence itself, or the force; this also has been estimated without any reference to the path along which it impels the body; but it should be observed, that a force, when commencing its influence on a body, may find that body already in motion, and, as is easy to conceive, may act on it so as to divert it from its original path, and cause it to describe some other; in such a case the body may be moving under the influence of two forces, or under the influence of an impulse and a force; but still there must exist, or at least we can conceive, some single force which if immediately applied to the body, at any instant of time, would give it the same motion that it actually has at that instant in virtue of the combined influences alluded to. Now it must be remembered that it is this

single and equivalent force which F represents in the foregoing equations, and which, when its intensity is the same, is always measured by the same linear space or length of path, be this path whatever it may. By the path of a body, urged by an accelerative force, is meant the track of its centre of gravity.

Another circumstance of importance deserves to be mentioned here, viz. that the general expression $\frac{ds}{dt}$ for the velocity at any point of the path is no other than the differential coefficient of the variable path s taken relatively to the independent variable t. It is from this circumstance, as we shall hereafter see, that we are enabled to determine the path of a moving body from knowing its velocity at any point of it in quantity and direction. In like manner the general expression for the force is the differential coefficient of the velocity taken relatively to the same independent variable, and this expression combined with that for the velocity, leads, as we shall presently see, (equa. D,) to the expression $F=v\frac{dv}{ds}$, which is sufficient to determine the force which influences the body, when we know what function the velocity v is of the space s.

(106.) It will be expedient, for the convenience of reference, to collect together here the fundamental equations of motion now established, introducing such slight modifications of form as may tend to facilitate their practical applications in our future inquiries, and deducing from them such brief inferences as may be of more especial interest or importance. It will be best to keep distinct those equations which refer to constant forces, or to motion uniformly-accelerated, from those which refer to variable forces, or to motion not uniformly accelerated.

I. *When the accelerating Force is constant.*

Referring to equation (2) we have

$$dv=F\,dt \;\therefore\; v=Ft+c \;\ldots\ldots\; (A).$$

If t become 0 when v does, that is, if t is measured from the commencement of motion, the constant c vanishes, and we infer from this expression for v, that *the velocities acquired in any times, reckoning from the commencement of motion, are proportional to the times themselves.*

Introducing the value $v=Ft$ in the equation (4), we have

$$ds=Ft\,dt \;\therefore\; s=\tfrac{1}{2}Ft^2=\tfrac{1}{2}vt \;\ldots\ldots\; (B),$$

no constant being added, because s vanishes with t. From this equation we infer, that *the spaces measured from the commencement of motion are proportional to the squares of the times.* We

may further remark here, that if the acquired velocity $v=Ft$ were to continue uniform during the time $\frac{1}{2}t'$, the space passed over in that time would be, (100), $s=\frac{1}{2}Ft^2$; hence the space described from the commencement of motion is equal to that which would be described in half the time by the body moving uniformly with the acquired velocity.

If we eliminate t by means of the equations (A), (B), disregarding the constant c, we shall have

$$v^2=2Fs \therefore v=\sqrt{2Fs} \dots . (C);$$

showing that the *spaces described from the commencement are proportionals to the squares of the acquired velocities.*

The foregoing equations are, obviously, sufficient to determine any two of the quantities F, t, s, v, when the other two are given, the time being supposed to be reckoned from the beginning of the motion. But when this is not the case, and the time is supposed to commence not till the body has acquired a given velocity v_1, then regard must be had to the constant in (A); the value of this constant is plainly $c=v_1$, because by hypothesis v_1 is what v becomes when $t=0$. Equation (A) will, therefore, here be $v=Ft+v_1 \dots . (A')$; equation (B) will be $s=\frac{1}{2}Ft^2+v_1t \dots . (B')$; and by eliminating t from these two, we have for (C) the equation

$$v^2=2Fs+v_1^2 \therefore v\sqrt{2Fs+v_1^2} \dots . (C').$$

II. *When the accelerating Force is variable.*

From equations (4) and (2) we have

$$v=\frac{ds}{dt},\ F=\frac{dv}{dt} \therefore \frac{F}{v}=\frac{dv}{ds} \therefore Fds=vdv$$

$$\therefore \int F\,ds=\tfrac{1}{2}v^2 \dots . (D);$$

which equation is sufficient to determine the velocity v when we know what function the force F is of the space s, or it is sufficient to determine this function when we know the function v. From the same equation, also,

$$s=\int\frac{v}{F}dv \dots . (E);$$

which makes known the space described when we know what functions v and F are of this space.

And, lastly, from the equation (4) $t=\int\frac{ds}{v} \dots . (F)$; which determines the numerical value of t.

Having established these equations, we shall now proceed to exhibit their practical application, more especially to those motions which are presented to us in nature.

CHAPTER II.

ON THE RECTILINEAR MOTION PRODUCED BY A CONSTANT FORCE.

(107.) The most remarkable and important instance of the action of a constant force is that which nature presents to us in what we have called *gravity*, being that force, in virtue of which all bodies near the earth fall to its surface, with a uniformly accelerated velocity, in a vertical direction.

Numerous and very accurate experiments have fully established the fact, that the velocity of a falling body, when all resistance is removed, *is* uniformly accelerated, and that its direction *is* that of a vertical line, or a normal, to the earth's surface at the point where it falls. Such experiments, however, made at any particular place on the motions of bodies falling from a small elevation, are not sufficient to warrant the conclusion that gravity is really a constant force in the acceptation in which we use the expression. All that we can fairly infer from them is that, at the same place, and within the range of small elevations, no sensible variation of force is discoverable, and that, therefore, within the limits of our experiments, at least, gravity may be considered as a constant force. But to ascertain the real nature of gravity, by means of such experiments as these, it is obvious that they ought to be repeated in various parts of the earth, and at great elevations as well as small. This indeed has accordingly been done, and it has been always found that a heavy body carried to the summit of a high mountain loses part of its weight, shewing, therefore, that gravity acts with less intensity at the summit than at the base of the mountain;* and, on the contrary, it has always been found that the body increases its weight when carried into those latitudes which are nearer to the centre of the earth. These results of observation are doubtless sufficient to show that gravity is not a constant force, and, moreover, that its variation depends, in some way, upon the distance from the centre at which it acts. But it is doubtful, chiefly on account of the comparatively small elevations attainable by man, and partly on account of the im-

* At an elevation of a mile above the surface of the earth, the intensity of gravity is diminished $\frac{1}{1977\cdot 291}$ part; and a pendulum clock, beating seconds at the level of the sea, would lose 21·898 seconds a day at this altitude, a quantity not to be overlooked. Any traveller, having leisure and the proper apparatus, might try the experiment in the barrack on Mont Cenis, or at the Hospice of St. Bernard.—*Herschel's Physical Astronomy. Ency. Met.*

perfection of instruments, whether from such experiments the real law of the variation of gravity could have ever been safely inferred. The discovery of this law, as well, indeed, as of that which retains the planets in their orbits, was in fact the result, not of experiment, but of conjecture; but then it was the conjecture of NEWTON.

He was the first who conceived the splendid idea, and who afterwards fully verified and established the important fact, that the attractive force, not only of the earth but of every body in the solar system, decreases in intensity in the same proportion as the square of the distance from the centre of the attracting body increases. This, therefore, is the law of universal gravitation, and which, as Sir John Herschel beautifully observes, governs equally "the fall of a leaf and the precession of the equinoxes."

The investigation of this law is not fitted for this place; it belongs indeed, to Physical Astronomy, but we propose to touch upon it hereafter, at present we confine our attention to those motions which take place near enough to the surface of the earth to render the variation of gravity inappreciable. We shall shortly see that the expression for the force of gravity at the earth's surface is about 32 feet, and, from the observation in the note, it appears that at a mile above the surface this value is diminished only by about the 2000th part, which is too small to affect sensibly the circumstances of the motion of a falling body computed on the hypothesis that the force suffers no variation at all.

On the vertical Motion of heavy Bodies.

(108.) Let g represent the force of gravity, then, for the space descended by a heavy body in t seconds, we have by (B) the expression, $s=\frac{1}{2}gt^2$; and consequently, the space descended in one second is $s=\frac{1}{2}g$. Now this space has been ascertained, by very accurate experiments, to be in the latitude of London $16\frac{1}{12}$ feet, very nearly;* hence

$$16\tfrac{1}{12}\text{ ft.}=\tfrac{1}{2}g \;\therefore\; g=32\tfrac{1}{6}\text{ ft.};$$

this, therefore, is the expression for the force of gravity at the earth's surface, and in vacuo.

1. To determine the space through which a heavy body will descend in four seconds at the latitude of London, and also the velocity it will acquire.

* From the most recent experiments in the latitude of London, the value of g is found to be 193·14 inches, which is rather greater than $32\frac{1}{6}$ feet, this latter being indeed the value of gravity at about the latitude of 45°. The number $32\frac{1}{6}$ is, however, still retained in most of our elementary books, and will serve equally well for the purposes of practical illustration.

Using g for F the expression (B) gives for the space

$$s=\tfrac{1}{2}\,gt^2=16\tfrac{1}{12}\times 4^2=257\tfrac{1}{3}\text{ ft.};$$

also the equation (A) gives for the velocity $v=gt=32\tfrac{1}{6}\times 4=128\tfrac{2}{3}$ f

2. To determine in what time a heavy body will descend 40 feet.

$$\text{From (B) } t=\sqrt{\frac{2s}{g}}=\sqrt{\frac{800}{32\tfrac{1}{6}}}=4\frac{76}{77};$$

hence the time is $4\frac{76}{77}$ seconds.

3. If a body be projected downwards, with a velocity of 30 fee in a vertical direction, how far will it fall in four seconds?

By equation (B') $s=\frac{1}{2}\,gt^2+v_1\,t=16\tfrac{1}{12}\times 16+30\times 4=377$ feet.

4. A body is projected vertically upward with a velocity of 12 feet, how high will it ascend in 3 seconds?

Here since gravity *retards* the motion of the body, it must be considered as negative, and we have, from equation (B')

$$s=-\tfrac{1}{2}\,gt^2+v_1\,t=-16\tfrac{1}{12}\times 9+120\times 3=215\tfrac{1}{4}\text{ feet.}$$

5. To what height above the surface of the earth will a body ascend which is projected vertically upward with a velocity of 100 feet?

It will, obviously, ascend to the same height that it must fall from, to acquire a velocity of 100 feet; hence, from equation (A),

$$v=gt\ \therefore\ t=\frac{100}{32\tfrac{1}{6}}=3\cdot 11;$$

and from equation (B), $s=\frac{1}{2}\,v\,t=155\frac{1}{2}$ feet.

7. With what velocity must a body be projected to reach a height of 579 feet?

From equation (C) $v=\sqrt{\{2\,gs\}}=\sqrt{\{64\tfrac{1}{3}\times 579\}}=193$ feet.

8. With what velocity must a body be projected downwards from the top of a tower, whose height is 150 feet, so that it may arrive at the bottom in two seconds?

Calling the velocity v_1, equation (B') gives $s=\frac{1}{2}\,gt^2+v_1\,t$

$$\therefore v_1=\frac{s}{t}-\tfrac{1}{2}\,gt=\frac{150}{2}-16\tfrac{1}{12}\times 2=42\tfrac{5}{6}\text{ feet.}$$

9 Suppose a body is let fall from a height of 300 feet, and that two seconds afterwards another body is let fall from a height of 200 feet, in what time will the former overtake the latter?

Let us suppose that the second body will have been in motion x seconds when the first overtakes it, then the first will have been in motion $x+2$ seconds; consequently, the space described by the second will be $s=\frac{1}{2}\,gt^2=16\frac{1}{12}x^2$; and, therefore, the space described

by the first must be $16\frac{1}{12}x^2+100$, but this space is also $s=\frac{1}{2}gt^2=16\frac{1}{12}(x+2)^2$; consequently, $16\frac{1}{12}x^2+100=16\frac{1}{12}(x+2)^2$.

$$\therefore\ 100=64\tfrac{1}{3}x+64\tfrac{1}{3}\ \therefore\ x=\frac{107}{193};$$

hence they will meet $\frac{107}{193}$ of a second.

10. How far must a body fall to acquire a velocity of 90 feet?

Ans. 125·9 feet.

11. What space was described in the last second by a body which had fallen 7 seconds? *Ans.* $209\frac{1}{12}$ feet.

12. With what velocity must a body be projected into a well 350 feet deep, that it may arrive at the bottom in 4 seconds?

Ans. $23\frac{1}{8}$ feet.

On the Motion of Bodies along inclined Planes.

(109.) When a body is placed on an inclined plane the force of gravity produces a certain pressure, represented by the weight of the body: if we resolve this vertical pressure P in two directions, the one along the plane, and the other perpendicular to it, the former component will be P sin. i, taking i for the inclination of the plane to the horizon, and, to prevent the body from moving down, this is the force or pressure which must be counterbalanced. As, therefore, P represents the force of gravity in the vertical direction, and P sin. i the force in the direction of the plane, and moreover, as g represents the vertical acceleration, we shall have for the acceleration down the plane, (see p. 121,) P : P sin. i : : g : g sin. i; hence the body is urged down the plane by the constant force, $g'=g\sin.i$;

and, therefore, substituting in the formulas (106) this value of g' for F, they will then comprise the whole theory of motion down an inclined plane, whether the body have an initial velocity up or down the plane or not.

If l represent the length of the plane, and h its height, then $\sin.i=\frac{h}{l}$; hence the accelerating force is $g\frac{h}{l}$; and, therefore, the velocity acquired in descending down the whole length l, that is in descending through the space $s=l$, by the influence of this force must be (C), $v=\sqrt{\{2gh\}}$; which expression, being independent of l, shows that the velocity acquired in descending down all planes of the same height is equal to the velocity acquired in falling through that height.

The velocities of two bodies, the one falling through the perpendicular height, and the other falling through the length of the plane, are respectively $v=gt$, $v'=gt'$ sin. i; but, as these velocities are

equal, we must have $gt=gt'$ sin. i $\therefore$ $t=t'$ sin. i; so that the time of falling through the height is to the time of falling through the length, as sin. i to 1. But if we wish to know what extent of length is gone through by the one body, while the other goes through the whole height, then referring to the expressions for the spaces, we have

$$s=\tfrac{1}{2}gt^2=\tfrac{1}{2}gt'^2,\ s'=\tfrac{1}{2}gt'^2 \text{ sin. } i;$$

and these also are to each other as 1 to sin. i. If, therefore, from B (fig. 93) we draw the perpendicular BD, AD will be the length gone through by one body, while the other falls through the height AB, because AB : AD : : 1 : sin. ABD=sin. C=sin. i. If we draw the vertical DB′ and BB′ perpendicular to DB, then the time of falling through DB′ would equal the time of falling through DB, but DB′=AB, therefore the time of falling through AB is equal to the time of falling through either of the inclined planes AD, BB′. Hence this remarkable property of the circle, viz.: If from the extremities A, B, (fig. 94,) of the vertical diameter AB, cords be drawn, a body would fall through either of them in the same time that it would fall through the vertical diameter.

(110.) We shall now add an example or two of motion on an inclined plane.

1. The length of an inclined plane is 60 feet, and its inclination 30°, what velocity would a body acquire in falling down it for 2″?

Substituting, g sin. i, for F, in the equation (A), we have

$$v=gt \text{ sin. } i=32\tfrac{1}{6}\times 2\times\tfrac{1}{2}=32\tfrac{1}{6} \text{ feet.}$$

2. How long would a body be in falling down an inclined plane whose length is 100 feet, and inclination 60°?

Substituting g sin. i for F, in the equation (B), we have

$$t=\sqrt{\frac{2s}{g \text{ sin. } i}}=\sqrt{\frac{200}{32\frac{1}{6}\times\frac{1}{2}\sqrt{3}}}=\frac{10}{\sqrt{\{16\frac{1}{12}\times\frac{1}{2}\sqrt{3}\}}}=2\cdot6 \text{ seconds.}$$

3. If a body be projected up an inclined plane whose length is ten times its height, with a velocity of 30 feet, in what time will the velocity be destroyed?

The time is necessarily the same as would be required to produce a velocity of 30 feet in a body falling from rest down the same plane; hence making the substitution of $g\frac{h}{l}$ for g, in the equation (A), we have $t=\frac{vl}{gh}=\frac{30\times10}{32\frac{1}{6}\times1}=9\cdot3$ seconds.

4. A body is projected up an inclined plane whose height is $\frac{1}{8}$th of its length, with a velocity of 50 feet. Find its place, and the velocity, after 6″ have elapsed.

Here the force $g\frac{h}{l}$ *retards* the motion of the body, and must, therefore, be considered as negative: hence, from equation (B′), we have

$$s=v_1 t-\tfrac{1}{2}g\frac{h}{l}t^2=50\times 6-16\tfrac{1}{12}\times\frac{1}{6}\times 36=203\tfrac{1}{2}\text{ feet,}$$

and from equation (A), $v=-g\frac{h}{l}t=-32\tfrac{1}{6}\times\tfrac{1}{6}\times 6=-32\tfrac{1}{6}$ feet;

$$\therefore\ 50-32\tfrac{1}{6}=17\tfrac{5}{6}\text{ feet, the velocity required.}$$

5. How long would a body be in falling down an inclined plane whose height is to its length as 7 to 15, to acquire a velocity of 20 feet? *Ans.* 1·3 seconds.

6. Required the length of a plane whose inclination is 30° that will cause a body let go at the top, to acquire a velocity of 500 feet when it reaches the bottom. *Ans.* 7772 feet.

It should be remarked, that in what is here said about motion along an inclined plane, friction is entirely disregarded; the body being supposed to slide freely down the plane without suffering the least impediment.

(111.) The two problems following are added as a further illustration of the motions of bodies under different modifications of gravity, and, also, as an additional application of the principle stated at (p. 121).

Problem I.—Two weights W, W_1, connected by a thread passing over a small pulley C, as in fig. 95, are placed upon the two inclined planes CA, CB; to determine the circumstances of their motion.

The vertical pressure of the whole mass, produced by the force of gravity, is $W+W_1$; the acceleration which would be produced by the same force is g. Again, the pressure of W in the direction WA, or which is the same thing the tension of the thread, W_1 C, is W sin. i; also the pressure of W_1, in the direction W_1 B, is W_1 sin. i_1; hence the system must move in virtue of the difference of these two pressures and to find with what acceleration F we have (p. 121,)

$$W+W_1 : W_1\sin. i_1-W\sin. i : g :: \frac{W_1\sin. i_1-W\sin. i}{W_1+W}g=F;$$

this, therefore, is the expression for the accelerative force which urges W_1 down the plane CB, and which, consequently, draws W up the plane AC; and, therefore, substituting this expression instead of F in the equations (A) and (B), at art. (106), we have, for the velocity acquired and space passed over at the end of t seconds, after the commencement of motion, the expressions

$$v=\frac{W_1 \sin i_1 - W \sin. i}{W_1+W} gt;\ s=\frac{W_1 \sin. i_1 - W \sin. i}{2(W_1+W)} gt^2.$$

If the two planes were vertical, then the problem would be to determine the motion when the two weights hang vertically at the ends of a thread passing over a pulley; since, therefore, in this case, sin. i and sin. i_1 are each unity, we have

$$\frac{F}{g}=\frac{W_1-W}{W_1+W},\ v=\frac{W_1-W}{W_1+W} gt,\ s=\frac{W_1-W}{2(W_1+W)} gt^2.$$

If only one of the planes were vertical, the problem would be to determine the motion when one weight W_1, hanging freely, draws another W up an inclined plane. In this case sin. $i_1=1$

$$\therefore F=\frac{W_1 - W \sin. i}{W_1+W} g.$$

If one of the planes were vertical and the other horizontal, the problem would be to determine the motion when W_1, hanging vertically, draws W along a horizontal plane. In this case sin. $i=0$

$$\therefore F=\frac{W_1}{W_1+W} g.$$

PROBLEM II.—A given weight W_1 is to draw another given weight W up an inclined plane of given height h; required the length l of the plane in order that the time of ascent may be the least possible.

The inclination of the plane being represented by i, as usual, we have sin. $i=\frac{h}{l}$, and the above expression for F may, therefore, be written

$$F=\frac{W_1 l - W h}{W_1 l + W l} g;$$

hence, by equation (B), the expression for s or the length l is

$$l=\tfrac{1}{2} F t^2=\frac{W_1 l - W h}{W_1 l + W l}\cdot\frac{t^2 g}{2}\ \therefore t=\sqrt{\frac{2(W_1+W) l^2}{(W_1 l - W h) g}};$$

this expresses the time when l as well as h is given. To determine, therefore, the value of this expression when a minimum, we must put the first differential coefficient derived from it, equal to 0, l being the independent variable; or, we may omit the radical, as also the constant factors before differentiating, (*Diff. Calc.* p. 8,) and we shall then only have to make

$$\frac{l^2}{W_1 l - W h}=\text{min.}\ \therefore \frac{W_1 l - W h}{l^2}=\frac{W_1}{l}-\frac{W h}{l^2}=\text{max.}$$

$$\therefore -\frac{W_1}{l^2}+\frac{2 W h}{l^3}=0\ \therefore l=\frac{2 W h}{W_1}.$$

On the Motions of Projectiles.

(112.) Although we do not intend to consider the general theory of curvilinear motion in the present section, yet it will be advisable to discuss here that particular case of it which we observe in bodies when projected obliquely into space, near the earth's surface. We know that every body so projected is influenced by two distinct causes, viz. the primitive impulsion of which the effect is to give the body some determinate and uniform velocity in a straight line, and the force of gravity, of which the effect is continually to draw down the body in a vertical direction; these verticals tend to the earth's centre, but throughout the path of a projectile they may without sensible error be considered as parallel. On this hypothesis, and abstracting for the present, as in the case of falling bodies, from the resistance of the air, we may easily determine the curve which the body describes. There are, indeed, two methods of solving very readily this problem; one method is first to express by means of horizontal and vertical co-ordinates the equation of the straight line which the impulsion would compel the body to describe, if gravity did not act, and then to diminish the ordinate y by the deflection which gravity would cause for the time t''. Thus, assuming the point of departure as the origin of the horizontal and vertical axes, we have, for the *initial direction* of the body, the equation

$$y=ax;$$

but, in the time t'' gravity diminishes this value of y by $\frac{1}{2}\,gt^2$,

$$\therefore\ y=ax-\tfrac{1}{2}\,gt^2\ \ldots\ldots (1).$$

If v_1 be the velocity of projection, v_1 will express the linear space which in the absence of gravity the body would pass over in $1''$; hence in t'' it would pass over $v_1\,t$. Now the action of gravity being always *vertical*, it is obvious that this force cannot at all affect the *horizontal* advance of the moving body, so that, corresponding to any time t'', the *abscissa* will be the same whether gravity act or not; but, from what has just been said, this abscissa, in the absence of gravity, is $v_1\,t\cos.\theta$, θ being the angle of elevation of the piece; hence

$$x=v_1\,t\cos.\theta\ \therefore\ t=\frac{x}{v_1\cos.\theta}\ \ldots\ldots (2).$$

Substituting this value for t, in the equation (1), we have, for the equation of the path, $y=\tan.\theta.\,x-\dfrac{gx^2}{2\,v_1^{\,2}\cos.^2\theta}$. (3); which shows that the path of the projectile is a parabola, and that the rectangular axes are parallel to those of the curve (*Anal. Geom.* p. 183), so that the vertex of the parabola is the highest point of it.

If h denote the height due to the velocity v_1, that is to say, the height from which a body must fall vertically to acquire this velocity, then since (C) $v_1=\sqrt{\{2\,gh\}}$ the equation may be written

$$y=\tan.\,\theta\,x-\frac{1}{4\,h\cos.^2\theta}x^2 \quad \ldots\ldots (4)$$

The other method of obtaining the equation of the path to which we have alluded is this. Taking the same origin as before, let the direction of projection be taken for the axis of y, and a vertical line drawn downwards for the axis of x; then v_1 being the initial velocity, as before, we have

$$x=\tfrac{1}{2}\,gt^2,\ y=v_1\,t=t\sqrt{\{2\,gh\}} \quad \ldots\ldots (1);$$

h being the height due to the initial velocity.

Eliminating t we get $y^2=4\,hx$ .... (2); the equation of the parabolic path, and from which it appears that h is the distance of the origin, or point of projection, from the focus of the parabola,* and as this is equal to the distance of the same point from the directrix, it follows from equation (1) that the velocity at any point of the curve is equal to the velocity acquired in falling vertically from the directrix to that point. Having thus determined the nature of the path of a projectile, we shall now subjoin a few general problems arising out of this determination.

Problem I.—(113.) To determine the angle of elevation θ, for which the *range* AB may be the greatest possible.

The general expression for the range or horizontal distance is the value of x, given by equation (4) for $y=0$, that is, it is

$$x=4\,h\cos.^2\theta\tan.\,\theta=2\,h\sin.2\,\theta \ldots\ldots (1);$$

and as this expression is to be a maximum, we must have

$$\frac{dx}{d\theta}=4\,h\cos.2\,\theta=0\ \therefore\ \theta=45^\circ \ldots\ldots (2);$$

which gives from (1) $x=2\,h$. (2), for the greatest range.

As $\sin.2\,\theta=\sin.2\,(90^\circ-\theta)$, it follows from the general expression (1) for the range, that the range is the same for $90^\circ-\theta$ as for θ, that is the ranges are the same whether the initial direction forms an angle below the line of 45° or an equal angle above it, (fig. 96.)

Problem II.—Knowing the range of a shot with a given charge of powder and a given elevation of the piece, to determine the range at any other elevation.

Suppose we know the maximum range R, or that due to the elevation of 45°, then from equation (2), above, the height due to the velocity of projection, is $h=\frac{1}{2}$ R; hence this is the value of h, for all

* Any point in the path may, obviously, be considered as the point of projection.

elevations with the same charge. Calling, therefore, the range due to any other elevation θ, r the expression for its value, will be $r=$ R sin. 2θ.

Thus any range is known by means of the maximum range. Or if we know any range r corresponding to the elevation θ, then to determine the range r' corresponding to another elevation θ', we have the two equations $r=\text{R sin. } 2\theta$, $r'=\text{R sin. } 2\theta'$,

to eliminate R; hence $\dfrac{r'}{r}=\dfrac{\sin. 2\theta'}{\sin. 2\theta}$ $\therefore r'=\dfrac{\sin. 2\theta'}{\sin. 2\theta}r$.

PROBLEM III.—Given the angle of elevation and the initial velocity, to determine the time of flight, and the greatest height of the projectile.

Returning to equation (1), art. (112), we have, when $y=0$,

$$t=\sqrt{\frac{2\,ax}{g}}=\sqrt{\frac{2\tan.\theta.\,x}{g}},$$

but, by problem I., $x=4h\cos.^2\theta\tan.\theta$; hence by substitution

$t=2\sin.\theta\sqrt{\dfrac{2\,h}{g}}$ . . . . (1); which expresses the time of flight.

To determine the greatest height above the horizontal plane we must find the maximum value of y, from equation (4) art. (112), for which purpose we have the equation

$$\frac{dy}{dx}=\tan.\theta\frac{x}{2\,h\cos.^2\theta}=0 \;\ldots\ldots (2)$$

$$\therefore x=2\,h\cos.^2\theta\tan.\theta=h\sin. 2\theta \;\ldots\ldots (3);$$

which, by equation (1) prob. I., is half the whole range: putting this value for x in the equation of the curve, we have for y

$$y=2\,h\sin.^2\theta-h\sin.^2\theta=h\sin.^2\theta \;\ldots\ldots (4),$$

which expresses the greatest height.

If $\theta=45°$, $\sin.^2\theta=\frac{1}{2}$ $\therefore y=\frac{1}{2}h$, so that (prob. I.), the greatest height is one-fourth of the range.

The expression (2) denotes the tangent of the angle which the curve makes with a horizontal line at any point (x, y).

PROBLEM IV.—Given the initial direction to determine the velocity, so that the projectile may pass through a given point.

Let (x', y') be the given point, then by the equation of the curve (p. 131),

$$y'=\tan.\theta.\,x'-\frac{gx'^2}{2\,v_1^2\cos.^2\theta}\;\therefore v_1=\frac{x'}{\cos.\theta}\sqrt{\frac{g.}{2\,(\tan.\theta\cdot x'-y')}}.$$

PROBLEM V.—When the velocity of projection is given, to determine the direction so that the projectile may pass through a given point.

By substituting in equation 4, art. (112), $\sec.^2\theta$, or rather $1+\tan.^2\theta$, for

$$\frac{1}{\cos.^2\theta}, \text{ we have } y'=\tan.\ \theta.\ x'-\frac{1+\tan.^2\theta}{4h}x'^2$$

$$\therefore \tan.^2\theta-\frac{4h}{x'}\tan.\ \theta=-\frac{4hy'}{x'^2}-1; \text{ this quadratic solved}$$

for $\tan.\ \theta$ gives $\tan.\ \theta=\frac{2h\pm\sqrt{\{4h^2-4hy'-x'^2\}}}{x'}$ . . . . (1);

so that there are two different directions whenever the problem is possible, except when $4h^2=4hy'-x'^2$, or $(2h-y')^2=x'^2+y'^2$, in which case there is but one direction, but when

$$(2h-y')^2>x'^2+y'^2,$$

the problem becomes impossible under the proposed conditions.

The time elapsed from the instant of projection till the projectile reaches the proposed point is, by equation (1), art. (112),

$$t=\sqrt{-\frac{\{2y'+2\tan.\ \theta.\ x'\}}{g}} \quad . . . . (2).$$

PROBLEM VI.—To determine the range on an oblique line passing through the point of projection, and also the time of flight.

Let i be the inclination of the oblique line to the horizon, then its equation is $y'=\tan.\ i.\ x'$ . . . . (1); combining this with the equation of the projectile we shall obtain the abscissa of the point, where it meets this line, by the equation

$$\tan.\ i.\ x'=\tan.\ \theta.\ x'-\frac{x'^2}{4h\cos.^2\theta}$$

$$\therefore x'=4h\cos.^2\theta\,(\tan.\ \theta-\tan.\ i)=\frac{4h\cos.\ \theta\sin.\ (\theta-i)}{\cos.\ i} \quad . . . (2);$$

and, consequently, the oblique range will be

$$r=\frac{x'}{\cos.\ i}=\frac{4h\cos.\ \theta\sin.\ (\theta-i)}{\cos.^2 i} \quad . . . . (3);$$

and the time may be found from equation (2), last problem, by substituting for x' and y' the values (1) and (2) in this. This substitution gives $t=\sqrt{\left\{\frac{(\tan.\ \theta-\tan.\ i)\,8h\cos.\ \theta\sin.\ (\theta-i)}{g\cos.\ i}\right\}}$

$$=2\frac{\sin.\ (\theta-i)}{\cos.\ i}\sqrt{\frac{2h}{g}} \quad . . . . (4).$$

PROBLEM VII.—To determine the greatest range on an oblique plane, and the greatest height above it.

The angle of elevation, which belongs to the greatest range, will be that which renders the expression (3), last problem, a maximum, or, since i is constant, we must have

$$2\cos.\theta\sin.(\theta-i)=\max.$$
$$=\sin.\{\theta+(\theta-i)\}-\sin.\{\theta-(\theta-i)\}$$
$$=\sin.(2\theta-i)-\sin.i$$
$$\therefore \sin.(2\theta-i)=\max. \therefore 2\theta-i=90^\circ \therefore \theta=\tfrac{1}{2}(90^\circ+i).$$

Putting, therefore, this value of θ in the expression (3) for the range, we have for the maximum range R

$$R=\frac{4h\cos.\frac{1}{2}(90^\circ+i)\sin.\frac{1}{2}(90^\circ-i)}{\cos.^2 i}$$

$$=\frac{2h(1-\sin.i)}{1-\sin.^2 i}\text{(*Dr. Young's Trig.* art 26)}=\frac{2h}{1+\sin.i}$$

To determine the greatest height M′P above AB′ (fig. 97), we must make PM — MM′ a maximum, or, since MM′$=x\tan.i$,

$$M'P=\tan.\theta.x-\frac{x^2}{4h\cos.^2\theta}-\tan.i\,x=\max.$$

$$\therefore \tan.\theta-\frac{x}{2h\cos.^2\theta}-\tan.i=0$$

$$\therefore x=2h\cos.^2\theta(\tan.\theta-\tan.i)=\frac{2h\cos.\theta\sin.(\theta-i)}{\cos.i}.$$

This value of x being substituted in the above expression for M′P, which, since $\tan.\theta-\tan.i=\dfrac{\sin.(\theta-i)}{\cos.\theta\cos.i}$, is the same as

$$\frac{x}{\cos.\theta}\left\{\frac{\sin.(\theta-i)}{\cos.i}-\frac{x}{4h\cos.\theta}\right\}$$

gives for M′P, when a maximum, the value

$$M'P=\frac{2h\sin.(\theta-i)}{\cos.i}\left\{\frac{\sin.(\theta-i)}{\cos.i}-\frac{\sin.(\theta-i)}{2\cos.i}\right\}=\frac{h\sin.^2(\theta-i)}{\cos.^2 i}$$

(114.) Collecting together the principal results of the preceding propositions, we have the following formulas:

I. *When the Plane is horizontal.*

$$\text{Range}=2h\sin.2\theta,\ \text{time}=2\sin.\theta\sqrt{\frac{2h}{g}}$$

Greatest range $=2h$

Greatest height $=h\sin.^2\theta$.

II. *When the Plane is oblique.*

$$\text{Range}=4h\frac{\cos.\theta\sin.(\theta-i)}{\cos.^2 i}$$

$$\text{Time}=2\frac{\sin.(\theta-i)}{\cos.i}\sqrt{\frac{2h}{g}}$$

$$\text{Greatest range} = \frac{2h}{1+\sin. i}$$

$$\text{Greatest height} = \frac{h \sin.^2 (\theta - i)}{\cos.^2 i}.$$

These equations contain the whole theory of projectiles *in vacuo*; they may all be deduced, independently of analysis, by the aid of common geometry, and a few well known properties of the parabola. See the second volume of *Dr. Hutton's* Course of Mathematics.

CHAPTER III.

ON THE RECTILINEAR MOTION, PRODUCED BY A VARIABLE FORCE.

(115.) We shall now proceed to show the application of the general formulas, at art. (106), to cases of rectilinear motion, produced by forces varying in intensity according to some known law. This variation is generally according to some function of the distance of the moving body from the fixed point, which is regarded as the centre of force, although, in some cases which nature presents, the variation is also dependent upon other circumstances; as, for instance, when the motion takes place, not in free space, but in a resisting medium, where, it is obvious, the body will be hindered from obeying the full influence of the attracting force by a resisting force, varying in some manner with the velocity. These particulars will be considered in prob. III.

Problem I.—(116.) To determine the vertical motion of a heavy body towards the earth; the force of gravity varying inversely as the square of the distance from the centre.

Call the radius of the earth r, the distance of the body from the centre at the commencement of motion a, and the distance at any time t'', after the commencement x; then by the hypothesis the intensity of the force F, at the time t'', will be given by the proportion

$$\frac{1}{r^2} : \frac{1}{x^2} :: g : \mathrm{F} \therefore \mathrm{F} = \frac{r^2}{x^2} g = -\frac{d^2x}{dt^2}.*$$

Having got an expression for the force, the next object is to deduce that for the velocity. Referring to equation (D), we have $v^2 = 2\int \frac{-r^2 g\, dx}{x^2} = \frac{2r^2 g}{x} + \mathrm{C}$; the constant C will depend upon

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* See note in next page.

the initial velocity of the body, that is, upon the velocity which it has at the distance a, where gravity begins to act; if this velocity is 0, then $\frac{2r^2 g}{a}+C=0$ $\therefore$ $C=-\frac{2r^2 g}{a}$; and thus the velocity of the body has at any time t'', that is, after having fallen from the distance a to the distance x is completely determined, it is $v^2=\frac{2r^2 g}{x}-\frac{2r^2 g}{a}=\frac{2r^2 g(a-x)}{ax}$; and when the body arrives at the surface of the earth, that is, when $x=r$, it will have acquired a velocity expressed by

$$v=\sqrt{\frac{2rg(a-r)}{a}};$$ which, if a is infinite, since $\frac{a-r}{a}$ is then 1, becomes $v=\sqrt{2rg}$; so that the velocity can never be so great as this, however far the body may fall, and, hence if it were possible to project a body vertically upwards with this velocity, it would go on to infinity and never stop. Of course this is on the supposition that there is no resisting medium nor other disturbing force. Taking the radius of the earth at 3965 miles, the last expression for v will be $v=6{\cdot}9506$ miles; so that if a body were to be projected upwards, with a velocity of about seven miles a second, and were to experience no resistance, it would never return to the earth.

It remains now to determine the time t''; and for this purpose we have the equation $-\frac{dx}{dt}$* $=v=r\sqrt{\frac{2g(a-x)}{ax}}$; from which we get

$$-dt=\frac{\sqrt{a}}{r\sqrt{2g}}\sqrt{\frac{x}{a-x}}\,dx=\frac{\sqrt{a}}{r\sqrt{2g}}\frac{x}{\sqrt{\{ax-x^2\}}}dx$$

$$\therefore t=\frac{\sqrt{a}}{r\sqrt{2g}}\int\frac{-x}{\sqrt{ax-x^2}}dx;$$

this integral may be immediately found by means of the general ex-

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* In the expression $\frac{ds}{dt}$ for the velocity at (p. 123), s is the space passed through; but here $a-x$ is the space, and $\frac{d(a-x)}{dt}=-\frac{dx}{dt}$, therefore, in this case, $v=-\frac{dx}{dt}$, and $\frac{dv}{dt}=F=-\frac{d^2x}{dt^2}$, and, generally, whenever F tends to diminish the space s, the increment Δs being always negative, the expression $\frac{ds}{dt}$, for the velocity, as also the expression $\frac{d^2s}{dt^2}$ for the force must, obviously, be likewise negative.

pression at the top of page 46 in the *Integral Calculus*, or we may proceed thus: to the numerator of the expression to be integrated add $\frac{1}{2}\,adx$, and then subtract the same quantity from it, and we shall thus convert the differential into two others, of which one will be immediately integrable by the rule for powers; thus we shall have the two expressions $\frac{\frac{1}{2}\,a\,dx - x\,dx}{\sqrt{\{ax-x^2\}}} - \frac{\frac{1}{2}\,adx}{\sqrt{\{ax-x^2\}}}$; the integral of the first of these is obviously $\sqrt{\{ax-x^2\}}$; that of the second is the elementary integral, $\frac{1}{2}\,a$ coversin.$^{-1}\,\frac{2}{a}\,x$ (*Int. Calc.* p. 10), or, which is the same thing, $\frac{1}{2}\,a\cos.^{-1}\left(\frac{2x-a}{a}\right)$. Consequently

$$t = \frac{\sqrt{a}}{r\sqrt{2g}}\left\{\sqrt{ax-x^2} + \tfrac{1}{2}\,a\cos.^{-1}\left(\frac{2x-a}{a}\right)\right\};$$

which expresses the number of seconds elapsed in moving from the distance a to the distance x, from the centre of attraction. This expression needs no correction, because $a = x$, and t become 0 at the same time. When $x = r$, that is, when the body arrives at the surface of the earth, the number of seconds elapsed will be

$$t = \frac{\sqrt{a}}{r\sqrt{2g}}\left\{\sqrt{ar-r^2} + \tfrac{1}{2}a\cos.^{-1}\left(\frac{2r-a}{a}\right)\right\};$$

which is evidently infinite, when a is, although the velocity, as we have before seen, is finite.

If the attracting body be considered as merely a point, then x may at length become 0, and the whole time of falling to the centre from the distance a will be expressed by

$$t = \frac{\sqrt{a}}{r\sqrt{2g}} \times \tfrac{1}{2}\,a\cos.^{-1} - 1 = \frac{\pi}{2r\sqrt{2g}}\,a^{\frac{3}{2}}.$$

If the body fall from any other distance a' the expression for t would be similar, so that $t^2 : t'^2 :: a^3 : a'^3$, that is, *the squares of the times of falling from rest to the centre of force are as the cubes of the distances from which they fall.*

PROBLEM II.—(117.) To determine the motion of a body attracted towards a fixed centre, the force varying directly as the distance.

This is the law of force which would attract a material point at liberty to move along a perforation from the surface to the centre of the earth. For the universal law of nature being this, that every particle of matter attracts with a force varying in intensity inversely as the square of the distance at which it acts, it follows that the attractive forces of homogeneous spheres must vary directly as their

masses, or as the cubes of their radii, and inversely as the squares of the distances of their centres from the attracted point. Now it is shown, by writers on Physical Astronomy, that the attraction of a sphere is the same as if its entire mass were concentrated in its centre; it is shown, moreover, that a particle placed any where within a spherical shell will remain at rest, being equally attracted in all directions. If, therefore, a particle be placed below the surface of the earth, and at the distance x' from the centre, it will be moved only by the force which resides in the inner sphere of radius x, as it is kept at rest by the influence of the shell, whose thickness is $r-x$; hence, from what has just been said,

$$\frac{r^3}{r^2} : \frac{x^3}{x^2} :: g : \mathrm{F} = \frac{g}{r} x;$$

which shows that the force varies directly as the distance x. Having thus got the value of F we have, as in last problem,

$$\tfrac{1}{2} v^2 = \int -\frac{g}{r} x \, dx = -\frac{g}{2r} x^2 + \mathrm{C}.$$

If the body be merely dropped into the hole, v will be 0, when $x=r$,

$$\therefore 0 = -\frac{g}{r} r^2 + 2\,\mathrm{C} \therefore \mathrm{C} = \tfrac{1}{2} gr \therefore v = \sqrt{\frac{g}{r}(r^2 - x^2)} \cdot (1);$$

which is the velocity of the body at any distance x from the centre when the body reaches the centre, that is, when $x=0$, the velocity is $v=\sqrt{\{gr\}} \cdots (2)$. This velocity must be spent before the body will stop, and as the motion after passing the centre will be retarded according to the same law, as it was before accelerated, the body will continue to move till it reaches the opposite point of the earth's surface, where it would stop; but being again attracted by the same force as at first, it will return and pass through the centre to the point of departure, and will thus move backwards and forwards continually.

To determine the time from the departure of the body till its arrival at the distance x from the centre, we have, as usual,

$$-\frac{dx}{dt} = v = \sqrt{\frac{g}{r}(r^2 - x^2)} \therefore dt = -\sqrt{\frac{r}{g}} \cdot \frac{dx}{\sqrt{\{r^2 - x^2\}}}$$

$$\therefore t = \sqrt{\frac{r}{g}} \times - \int \frac{dx}{\sqrt{\{r^2 - x^2\}}} = \sqrt{\frac{r}{g}} \times \cos.^{-1} \frac{x}{r};$$

which requires no correction, since $t=0$, when $x=r$. For the time of reaching the centre we have, by making $x=0$,

$$t = \frac{\pi}{2} \sqrt{\frac{r}{g}} \cdots (3).$$

By making $x = -r$ we have, for the time of passing through the

whole diameter $t = \pi\sqrt{\frac{r}{g}} \cdots (4)$; which is twice the time of falling to the centre as it ought to be.

If the body do not begin to move from the surface of the earth, but from some point within it at the distance r', instead of r, from the centre then $\frac{g}{r}r'$, will be the force at the commencement of motion instead of g, and, therefore, substituting this for g, and r' for r, the expression for t will be $t=\sqrt{\frac{r}{g}}\times\cos.^{-1}\frac{x}{r'}$; and, consequently, when $x=r'$, we have for the time of reaching the centre

$$t=\frac{\pi}{2}\sqrt{\frac{r}{g}},$$

the same as the expression (3). Hence, at whatever point within the surface of the earth the body be placed, it will reach the centre in the same time.

In order to find this time, take the radius r, of the earth, equal to 3965 miles, and we shall have $t''=21'7''\frac{1}{4}$, which will be the time occupied in passing to the centre, however near to it the body be placed.

It is obvious that if g represented the energy of any other force, instead of gravity, at the distance r from the centre, the reasoning and conclusions would be the very same, so that when a body is attracted from a state of quiescence by any centre of force, varying in intensity directly as the distance, the whole time of passing to the centre will be the same from whatever point the motion commences, whether from a point infinitely distant, or from a point infinitely near. Hence the body would pass over an infinite space in a finite portion of time, but then, by hypothesis, the force at the commencement of motion must be infinitely great.

We shall now consider the motion of a body near the surface of the earth, taking into account the resistance of the air, which we have hitherto neglected.

PROBLEM III.—(118.) To determine the vertical motion of a heavy body near the earth's surface, considering the resistance of the air to vary as the square of the velocity.

If we represent the resisting force at any time t'', after the commencement of motion by f, and the velocity generated by v, then, by hypothesis $f=mv^2$, m being constant for all velocities, and which can be determined only from experiment. Hence the force F, accelerating the body, is $F=g-mv^2$; so that here we have F as a function of the velocity, and not of the space as heretofore; therefore, since

$$\text{(D) } F=\frac{dv}{dt}\text{, we have } \frac{dv}{dt}=g-mv^2 \therefore dt=\frac{dv}{g-mv^2}.$$

The second member of this equation may be integrated by the method of rational frictions; or we immediately see that the expression is the same as $\left(\frac{dv}{g^{\frac{1}{2}}+m^{\frac{1}{2}}\,v}+\frac{dv}{g^{\frac{1}{2}}-m^{\frac{1}{2}}v}\right)\times\frac{1}{2\,g^{\frac{1}{2}}}$; consequently,

$$t=\left(\int\frac{dv}{g^{\frac{1}{2}}+m^{\frac{1}{2}}\,v}+\int\frac{dv}{g^{\frac{1}{2}}-m^{\frac{1}{2}}\,v}\right)\times\frac{1}{2\,g^{\frac{1}{2}}}$$

$$=\frac{1}{2m^{\frac{1}{2}}\,g^{\frac{1}{2}}}\{\log.\,(g^{\frac{1}{2}}+m^{\frac{1}{2}}\,v)-\log.\,(g^{\frac{1}{2}}-m^{\frac{1}{2}}\,v)\}+C$$

$$=\frac{1}{2\sqrt{mg}}\log.\,\frac{g^{\frac{1}{2}}+m^{\frac{1}{2}}v}{g^{\frac{1}{2}}-m^{\frac{1}{2}}v}\;\ldots\ldots(1);$$

the constant being 0, because the time and the velocity begin together.

This determines the time of the motion necessary to generate a given velocity. To find the velocity when the time is given it will be necessary to disengage the equation from logarithms, putting, therefore, e for the hyperbolic base, we have, since $\log.\,e=1$,

$$2\,t\sqrt{mg}\;\log.\,e=\log.\,\frac{g^{\frac{1}{2}}+m^{\frac{1}{2}}\,v}{g^{\frac{1}{2}}-m^{\frac{1}{2}}\,v}\;\therefore\; e^{2t\sqrt{mg}}=\frac{g^{\frac{1}{2}}+m^{\frac{1}{2}}\,v}{g^{\frac{1}{2}}-m^{\frac{1}{2}}\,v}\,.\;(2);$$

from which v may be found when t is known.

Since e exceeds unity, the first member of this equation increases with t, and is infinite when t is, consequently, as t approaches to infinity the denominator $g^{\frac{1}{2}}-m^{\frac{1}{2}}\,v$; must approach to 0, but when it is actually $0, v=\sqrt{\frac{g}{m}}$. (3); so that the longer the body is in motion, that is, the greater the space through which the body moves in a medium, varying in resistance as the square of the velocity, and, towards an attractive force, varying inversely as the square of the distance, the nearer will the velocity approach to constancy.

Having found the velocity we may readily determine the space s, through which the body has passed to acquire it. For, by (D),

$$vdv=Fds=(g-mv^2)\,ds.$$

$$\therefore s=\int\frac{vdv}{g-mv^2}=-\frac{1}{2m}\log.\,(g-mv^2)+C.$$

To determine C put $s=0$, then $v=0$,

$$\therefore 0=-\frac{1}{2m}\log.\,g+C\;\therefore\;C=\frac{1}{2m}\log.\,g$$

$$\therefore s=-\frac{1}{2m}\log.\,(1-\frac{mv^2}{g}).$$

If the space be already known the acquired velocity may be found by this equation.

If the body is projected with a velocity v, in the resisting medium in opposition to the force, then the motion becomes retarded, both gravity and the force mv^2 conspiring to stop the body; hence the *retardive* force is $F=-g-mv^2$

$$\therefore \frac{dv}{dt}=-g-mv^2 \therefore -dt=\frac{dv}{g+mv^2}$$

$$\therefore -t=\frac{1}{\sqrt{gm}}\{\tan.^{-1}\sqrt{\frac{m}{g}}.\,v\}+C.$$

We may determine C from the circumstance that at the commencement of motion, that is, when $t=0$, $v=v_1$, so that

$$0=\frac{1}{\sqrt{gm}}\{\tan.^{-1}\sqrt{\frac{m}{g}}v_1\}+C;$$

therefore, subtracting the former equation, from this, we have

$$t=\frac{1}{\sqrt{gm}}\{(\tan.^{-1}\sqrt{\frac{m}{g}}.\,v_1)-(\tan.^{-1}\sqrt{\frac{m}{g}}.\,v)\},$$

from which equation v may be determined for any proposed time t''. It remains now to deduce the expression for the corresponding space; for this purpose we must employ the formula (E), which gives

$$s=\int\frac{v}{F}\,dv \therefore -s=\int\frac{vdv}{g+mv^2};$$

that is, $-s=\frac{1}{2m}\log.\ (g+mv^2)+C$. As $s=0$, when $v=v_1$, we have

$$0=\frac{1}{2m}\log.\ (g+mv_1^2)+C$$

$$\therefore s=\frac{1}{2m}\{\log.\ (g+mv_1^2)-\log.\ (g+mv^2)=\frac{1}{2m}\log.\ \{\frac{g+mv_1^2}{g+mv^2}\}.$$

When the retardive force has destroyed the motion, that is, when $v=0$ we shall have $s=\frac{1}{2m}\log.\ \{1+\frac{mv_1^2}{g}\}$; which expresses the height to which the body will reach when projected with the velocity v. It will not acquire so great a velocity in returning to the earth, because the accelerating force is only $g-mv^2$.

SECTION II.

ON THE THEORY OF CURVILINEAR MOTION.

(119.) We now come to discuss the general theory of the motion of a *free point*, or material particle, independently of any restriction as to the nature of the path it is compelled to take.

We say *point* instead of *body*, because we do not propose to take into consideration, in the present section, any circumstances of the motion which may be dependent upon the mass of the moving body. Whenever, therefore, in the course of this section, we speak of the motion of a body, it must be noticed that we consider the acting forces to apply themselves equally to all the particles of the body, and that these particles exert no power themselves sufficient to modify the motion. This, indeed, is the hypothesis upon which the investigations in the preceding section are founded; where we have considered the motions of bodies chiefly in reference to the force of gravity.

But, in fact, as already hinted at (118), all bodies in nature exert a *mutual* influence on each other, and the intensity of this influence varies with the mass from which it emanates. Two bodies then, M and m, at liberty to obey their mutual attractions, approach each other in virtue of the force resident in M, combined with the force resident in m; and, therefore, the distance of their centres at any time will be the same as if the sum of the attractions due to M and m were combined in M, and, instead of the other body m, a single particle were placed at its centre; so that we should thus have to consider only the motion of a single particle or free point acted on by a single centre of force, viz. the centre of M's attraction. We may, therefore, after having attributed a proper value to the attractive force tending to M's centre, disregard the mass of the moving body m; and when, in the course of the present section, we speak of the motion of a body about a fixed centre of force, we consider the influence of the moving body to have been transferred to that centre as above, and, therefore, the body to move as a free point.

Let us now proceed to investigate the general equations of motion.

CHAPTER I.

ON THE GENERAL EQUATIONS OF THE MOTION OF A FREE POINT.

(120.) WHEN a material particle moves in a curve line, it is obvious that its direction at any point of its path is in the tangent at that point, along which, if it were there left to itself, it would proceed with a uniform velocity: and its curvilinear course is kept up only by the continual influence of some force or forces which at every instant deflect it from the rectilinear course it tends to pursue, in virtue of its inertia.

As far as effects are concerned we may consider a body thus moving to be impelled along the curve by an accompanying force, varying in intensity conformably to the circumstances of the motion, and we know that the value of this force at any time t'', during which the arc s has been described, will be expressed by

$$\frac{d^2 s}{dt^2}.$$

If this same force had a statical effect, or which is the same thing, if it were directly opposed by an equal force F, we might then, instead of the first force, substitute three others acting on the point and in the directions of three rectangular axes; if α, β, γ, be the angles which these form, with the line along which the point tends to move, then the values of these three forces will be

$$\text{F}\cos.\alpha,\ \text{F}\cos.\beta,\ \text{F}\cos.\gamma;$$

so that the point will have the same tendency to move under the influences of these three forces as under the influence of the original force F; these, therefore, are fitted to produce the same acceleration $\frac{d^2 s}{dt^2}$ as F; let us see what acceleration each alone is fitted to produce. In order to this let us first remark that since

$$\frac{dx}{ds}=\cos.\alpha,\ \frac{ds}{dt}\cos.\alpha=\frac{dx}{dt}\ \therefore\ \frac{d^2 s}{dt^2}\cos.\alpha=\frac{d^2 x}{dt^2}\ .\ (1);$$

then, by the principle stated at p. 121,

$$\text{F}:\text{F}\cos.\alpha::\frac{d^2 s}{dt^2}:\frac{d^2 s}{dt^2}\cos.\alpha=\frac{d^2 x}{dt^2}\ldots.(2);$$

hence the acceleration which F cos. α is fitted to produce, is $\frac{d^2 x}{dt^2}$, and, in like manner, the accelerations due to F cos. β and F cos. γ are $\frac{d^2 y}{dt^2}$, and $\frac{d^2 z}{dt^2}$; x, y, and z being the co-ordinates of the point

at the instant t''. Thus then the consideration of the curvilinear motion of a material point in space is reduced to the consideration of the rectilinear motions of its three projections along three rectangular axes, and which describe the rectilinear spaces $x, y, x,$ while the body itself describes the curve s. Calling the forces along these axes X, Y, and Z, we have

$$\frac{d^2 x}{dt^2}=\mathrm{X},\ \frac{d^2 y}{dt^2}=\mathrm{Y},\ \frac{d^2 z}{dt^2}=\mathrm{Z}\ \ldots.(3);$$

and these are *the general equations of the motion of a free point.*

The velocities of the projections are $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$, and the velocity of the point itself is $\frac{ds}{dt}$, but (*Diff. Calc.* p. 215,)

$$\frac{ds^2}{dt^2}=\frac{dx^2+dy^2+dz^2}{dt^2}\ \ldots.(4);$$

from which expression it follows that if the lines which represent the velocities $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, be taken for the edges of a rectangular parallelopiped, the diagonal of it will represent the velocity $\frac{ds}{dt}$ of the point in space; and it equally follows from the equation (2), combined with the two similar equations furnished by the other projections, that if the lines which represent the accelerative forces $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2}$, be taken for the edges, the diagonal will represent the accelerative force $\frac{d^2 s}{dt^2}$, which acts upon the point in space.

It thus appears that the velocity which a body actually has may always be decomposed into three, directed according to three rectangular axes; and the accelerative force, by which a body is actually influenced, may always be decomposed into three, directed according to three rectangular axes; and conversely such a system of velocities, or of forces, may be always compounded into one. When the motion is in a plane curve, one of the components is of course 0.

If we differentiate the equation (4), relatively to the independent variable t, we shall have

$$d\,\frac{ds^2}{dt^2}=2\,\frac{d^2 x\,.\,dx+d^2 y\,.\,dy+d^2 z\,.\,dz}{dt^2};$$

but, in virtue of equations (3), the second member of this equation is

$$2\,(\mathrm{X}\,dx+\mathrm{Y}dy+\mathrm{Z}dz);$$

consequently, returning to the integral, we have

$$(1)\ldots\ldots\frac{ds^2}{dt^2}=v^2=2\int(\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz)\ldots\ldots(2).$$

We infer, therefore, that when the component forces X, Y, Z, are known functions of the co-ordinates x, y, z, for every point of the body's path, or *trajectory*, as it is called, and when, moreover, these functions are such as to render $\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz$ an exact differential,* then the velocity of the moving body will be determinable from this equation. The complete integral of (2) will involve an arbitrary constant, which can only be determined from previously knowing the velocity at some known point of the trajectory, which is the same as if we knew the point at which the motion commenced.

Suppose the general integral of (2) were

$$v^2=f(x, y, z,)+\mathrm{C};$$

if we knew that the velocity, at the point $(a, b, c,)$, were v_1 we should have $v_1^2=f(a, b, c,)+\mathrm{C}$

$$\therefore\ v^2-v_1^2=f(x, y, z)-f(a, b, c).$$

From this equation it appears that when we know the functions that X, Y, Z, are of x, y, z, and the velocity at any point (a, b, c), we may find the velocity at any other point $(x, y, z,)$ merely from knowing its co-ordinates, without requiring to know either the form of the curve between the two points (a, b, c), (x, y, z), or the time of describing it; in bodies, therefore, which move in curves, returning into themselves, the velocity is always the same at the same point.

(121.) It is an important fact that the differential in equation (2) is always exact whenever the body moves under the influence of a force emanating from a fixed centre, or, indeed, when any number of fixed centres act on the body, provided always that the intensity of each force is a function of the distance of the point, or body, on which it acts.

Let there be but one such centre, then placing the origin of the co-ordinates there, and calling r the distance of the moving body from it, at any point of its path, we may express the force acting on it by $f\,r$, the form of the function f being known; and, by resolving this force according to the three axes, we have the components

$$\mathrm{X}=fr\,.\,\frac{x}{r},\ \mathrm{Y}=fr\,.\,\frac{y}{r},\ \mathrm{Z}=fr\,.\,\frac{z}{r}.$$

Substituting these expressions in the general equation (2) we have

* In order to this these functions must satisfy the conditions

$$\frac{d\mathrm{X}}{dy}=\frac{d\mathrm{Y}}{dx},\ \frac{d\mathrm{X}}{dz}=\frac{d\mathrm{Z}}{dx},\ \frac{d\mathrm{Y}}{dz}=\frac{d\mathrm{Z}}{dy},$$ (*See Int. Calc.* p. 174.)

$v^2 = 2\int f r \,.\, \frac{x\,dx + y\,dy + z\,dz}{r}$; but since $r^2 = x^2 + y^2 + z^2$ $\therefore$ $r dr = x\,dx + y\,dy + z\,dz$, therefore, by substitution, $v^2 = 2\int fr \,.\, dr$ . . . . (1); so that the velocity is always determinable.

The same may be shown generally as follows:

Having chosen the axes of reference, let the co-ordinates of one of the fixed centres be a, b, c, and let R be the force exerted by this centre on a point at the distance r from it. Now the cosines of the angles which r makes with the axes are, severally,

$\frac{x-a}{r}, \frac{y-b}{r}, \frac{z-c}{r}$; so that the three components of R are

$$R\frac{x-a}{r}, R\frac{y-b}{r}, R\frac{z-c}{r}.$$

In like manner, for the components of R_1, R_2, &c., the forces simultaneously acting on the point from a second, a third centre, &c., we have

$$\begin{array}{lll}
R_1\frac{x-a_1}{r_1}, & R_1\frac{y-b_1}{r_1}, & R_1\frac{z-c_1}{r_1} \\
R_2\frac{x-a_2}{r_2}, & R_2\frac{y-b_2}{r_2}, & R_2\frac{z-c_2}{r_2} \\
\vdots & & \\
R_n\frac{x-a_n}{r_n}, & R_n\frac{y-b_n}{r_n}, & R_n\frac{z-c_n}{r_n};
\end{array}$$

so that if we add together each of these three vertical columns of components, we shall have the expressions for X, Y, and Z, to be introduced into the general formula (2). But, before performing this addition, we should remark, that since

$$r_n^2 = (x-a_n)^2 + (y-b_n)^2 + (z-c_n)^2$$

$$\therefore r_n\,dr_n = (x-a_n)\,dx + (y-b_n)\,dy + (z-c_n)\,dz;$$

consequently, if we divide this equation by r_n, and multiply by R_n, and then put n successively equal to 0, 1, 2, &c., we shall have

$$\begin{aligned}
Rdr &= R\frac{x-a}{r}dx + R\frac{y-b}{r}dy + \frac{z-c}{r}dz \\
R_1\,dr_1 &= R_1\frac{x-a_1}{r_1}dx + R_1\frac{y-b_1}{r_1}dy + \frac{z-c_1}{r_1}dz \\
&\vdots \\
R_n\,dr_n &= R_n\frac{x-a_n}{r_n}dx + R_n\frac{y-b_n}{r_n}dy + \frac{z-c_n}{r_n}dz \\
\hline
\therefore Rdr + R_1\,dr_1 \ldots + R_n\,dr_n &= \quad X\,dx \quad + \quad Y\,dy \quad + Z\,dz;
\end{aligned}$$

hence, if R_n be a function of r_n whatever be n, the first member of this equation will be always integrable, and we shall have

$$v^2 = 2\int R\,dr + 2\int R_1\,dr \ldots . + 2\int R_n\,dr_n$$

(122.) We shall notice in this place a remarkable property connected with the motion of a body about a single centre of force, viz. that the trilineal spaces described by the *radius vector* or line joining the moving point and fixed centre are to each other as the times of describing them, and this whatever be the law according to which the intensity of the attractive force varies.

When a body acted upon by a single centre of force moves in a curve line, we may consider such motion to arise from a primitive impulsion given to the body which would alone have caused it to describe a straight line, but being continually acted upon by a force out of this line it is deflected from this path at the very commencement of motion, leaving it a tangent to the path it actually takes at the point of projection; as moreover nothing draws the body out of the plane in which the centre of force and line of projection are situated, the path of the body must be a plain curve. Hence, placing the origin of the rectangular axes at the centre of force S (fig. 98), and taking P for the place of the body at any time t'', we have

$$X = -F\cos.\alpha,\quad Y = -F\cos.\beta$$
$$= -F\frac{x}{r},\qquad = -F\frac{y}{r},$$

the negative signs being used because F tends to diminish the spaces x and y; consequently the equations of motion are

$$\frac{d^2x}{dt^2} = -F\frac{x}{r},\quad \frac{d^2y}{dt^2} = -F\frac{y}{r}.$$

To eliminate F multiply the first of these equations by y and the second by x, and subtract the products; there results

$$\frac{yd^2x - xd^2y}{dt^2} = 0 \ \ldots\ldots (1),$$

an expression altogether independent of the value of F, so that whatever results from it will hold even when F is repulsive.

Now it is easy to perceive that the numerator of this expression is the differential of $ydx - xdy$, therefore, multiplying by dt and integrating, we have $\frac{ydx - xdy}{dt} = C \ldots\ldots (2)$

$$\therefore \int ydx - \int xdy = Ct + C_1,$$
$$\text{or } 2\int dyx - xy = Ct + C^1 \ldots\ldots (3).$$

Let us inquire into the geometrical signification of the first member of this equation. The term $2\int ydx$ obviously expresses twice the area SP′PM, and the term xy is twice the triangle PSM, consequently the equation (3) is the same as

$$2 \text{ sector SP'P} = Ct + C_1.$$

Suppose t'' to commence when the body is at P′, then since when $t=0$, the sector $=0$ $\therefore C_1 = 0$, consequently

$$2 \text{ sector } SPP' = Ct \therefore \frac{\text{sector } SPP'}{t} = \tfrac{1}{2} C,$$

that is, as announced above, *the sectors described are proportional to the times of describing them, and therefore equal areas are described about S in equal times:* this remarkable and important property is called the *general principle of equal areas.*

(123.) In order to complete the theory of curvilinear motion, it may be as well to repeat here the general expression given at (103) for the force fitted to produce the motion which a body actually has at any instant if immediately urged by that force in the direction of its motion, that is, in the direction of a tangent to the curve at the point where it is at that instant. Calling such a force S, its value is

$$S = \frac{d^2 s}{dt^2} \text{ also } \tfrac{1}{2} v^2 = \int S\, ds.$$

The force S may be called the *tangential force;* we see from the second equation that upon this force the velocity of the body in its path wholly depends, and it is wholly expended in producing this velocity; if, therefore, all the forces which influence the body at any particular point were decomposed, each into two, one in the direction of the tangent, and the other in the direction of the normal, the sum of the former components, that is, the tangential force, would determine the velocity and direction of the body's motion at that point. It follows, therefore, that if among the forces which act upon a body there be any which always act in the direction of a normal to its path, the components of these must necessarily destroy each other in the expression (2), (p. 146,) for the velocity along the curve; because, as just observed, this velocity is wholly due to the tangential force.

(124.) Before closing this preliminary chapter, we shall briefly show how the equations of motion investigated in the preceding section, are to be deduced from the more general theory laid down in the present chapter.

As a first application, let it be required to determine the motion of a point moving from the effect of an impulsion only, then, as there is here no acceleration, the equations of the motion are

$$\frac{d^2 x}{dt^2} = 0, \quad \frac{d^2 y}{dt^2} = 0, \quad \frac{d^2 z}{dt^2} = 0,$$

multiplying each by dt and integrating, we have, for the velocities in the directions of the axes, the expressions

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c \quad \ldots \ldots (1),$$

and therefore (equation 4, p. 145,) the velocity along the path is

$v=\sqrt{\{a'^2+b'^2+c'^2\}}$ which is constant; that is, $\frac{ds}{dt}=C \therefore s=Ct+C_1$, C_1 being the space already passed over when t'' commences.

From equations (1) we can prove that the path of the body must necessarily be a straight line; for, multiplying each by dt and integrating, they give $x=at+a'$, $y=bt+b'$, $z=ct+c'$, by means of which eliminating t, and there results the following general relations among the co-ordinates, viz.

$$x=\frac{a}{c}z+\frac{a'c-ac'}{c},\ y=\frac{b}{c}z+\frac{b'c-bc'}{c}.$$

As a second application, let it be required to determine the motion of a projectile in space. Here the equations of the motion are $\frac{d^2x}{dt^2}=0$, $\frac{d^2y}{dt^2}=-g$, therefore, multiplying by dt and integrating, we have for the components of the velocity, $\frac{dx}{dt}=C$, $\frac{dy}{dt}=C'-gt$, multiplying by dt and integrating again, we have,

$$x=Ct,\ y=C't-\tfrac{1}{2}gt^2;\ \text{or, putting } C'=aC,$$
$$y=ax-\tfrac{1}{2}gt^2, \text{ as before found.}$$

CHAPTER II.

ON THE MOTION OF A BODY CONSTRAINED TO MOVE ON A GIVEN CURVE.

(125.) In this chapter we shall show the application of the foregoing general theory to the circumstances of constrained motion, the moving body being prevented from obeying the influence of the applied forces through the intervention of a rigid line.

When a material point is thus compelled to move on a curve, the curve offers at every point passed over a certain resistance to the motion in the direction of the normal, and it is in consequence of this resistance that the wonted path of the body is continually diverted and its motion confined to the curve. This resistance, therefore, may be considered as a normal force continually acting on the moving body, and which, combined with the other forces on the body, produces the motion which actually has place. Omitting the normal force and taking the components X, Y of the others on which alone the velocity along the curve depends (p. 149), we have, by equation (2, p. 146),

$$v^2=2\int(X\,dx+Y\,dy) \ldots . (1),$$

by means of which the velocity at any proposed point (x, y) of the

curve may be found, X and Y being functions of the co-ordinates. As shown at (p. 146), the expression after integration will take the form $v^2-v_1^2=f(x,y)-f(a,b)$, (a,b) being the point where the velocity is known to be v_1.

As this result is independent of the normal pressure, that is, as it remains the same whatever this pressure may be, we infer that it must remain the same whatever the curve between (a,b) and (x,y) may be, so that as long as the applied forces remains the same, and the velocity of the body at the point (a,b) remains the same, the velocity at any other point (x,y) will be the same by whatever path it arrives at it.

Precisely the same conclusions would follow if we had supposed the motion to be on a curve of double curvature instead of on a plane curve, as is very obvious from the equation at p. 146.

If the body move on the curve in virtue of an original impulse merely, then, since there are no acting forces, $X=0$ and $Y=0$, and consequently $v^2=v_1^2$, which shows that the primitive velocity will be preserved and continued unchanged whatever be the curve along which it moves.

Let us suppose the body to move down a curve in consequence of the action of gravity, then we have, by taking the axis of X vertical,

$$X=-g,\ Y=-0\ \therefore\ v^2=2\int X\,dx=v_1^2-2gx.$$

To determine v_1 let h be the height above the origin from which the body begins to descend, that is, the ordinate of the point at which the velocity is 0, then

$$0=v_1^2-2gh\ \therefore\ v_1^2=2gh\ \therefore\ v^2=2g(h-x)\ \ldots\ldots (2).$$

In the general case of this problem we have seen that the velocity depends on the co-ordinates x and y of the point arrived at, but is independent of the path to it; in this particular case we see that the velocity depends only on the ordinate x of the point arrived at, being independent both of the abscissa of the point and of the path, and thus *any point in a straight line parallel to the horizon will be arrived at with the same velocity if the body descend from a fixed point above it along any line or curve whatever*, the velocity being that which would be acquired by falling freely through the vertical height.

It immediately follows from this, that when the body has arrived at the lowest point of the curve, its acquired velocity will be sufficient to carry it up the ascending branch (if the curve have one,) to the same height as it descended from, whether the two branches be similar or not, although the times of descent and ascent will be different if the branches be different in form.

To determine the time requires that we know the curve of descent, in which case we have, by the general expression (D), art. (106), $v=$

$\frac{ds}{dt} \therefore dt = \frac{ds}{v}$, that is, in the case of gravity,

$$dt = \frac{ds}{\sqrt{\{2g(h-x)\}}} \therefore t = \int \frac{ds}{\sqrt{\{2g(h-x)\}}} \quad \ldots . (3).$$

(126.) We shall shortly give an example or two of this kind of constrained motion; but we shall first investigate a general expression for the resistance, or normal force, at any point of the constraining curve, when this curve is given.

Let APM (fig. 99,) be any given curve on which a material point P is compelled to move when acted upon by forces whose components are X and Y. Let PN represent the normal force or the resistance which the body receives when at P and call it R; the components of this force are R cos. NPC and — R cos. NPC′; consequently, taking into account all the forces which act upon the body, the equations of its motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} + \mathrm{R}\frac{dy}{ds} \\ \frac{d^2y}{dt^2} &= \mathrm{Y} - \mathrm{R}\frac{dx}{ds} \end{aligned}\right\} \quad \ldots . (1).$$

Multiplying the first of these by $\frac{dy}{ds}$, and, the second by $\frac{dx}{ds}$, and subtracting the second from the first, we have

$$\frac{dyd^2x - dxd^2y}{ds\,dt^2} = \mathrm{X}\frac{dy}{ds} - \mathrm{Y}\frac{dx}{ds} + \mathrm{R},$$

consequently, since (*Diff. Calc.* p. 136,) the general expression for the radius of curvature γ at any point (x, y) is

$$\gamma = \frac{ds^3}{dy\,d^2x - dx\,d^2y}, \text{ it follows that}$$

$$\mathrm{R} = \mathrm{Y}\frac{dx}{ds} - \mathrm{X}\frac{dy}{ds} + \frac{1}{\gamma}\frac{ds^2}{dt^2} = \mathrm{Y}\frac{dx}{ds} - \mathrm{X}\frac{dy}{ds} + \frac{v^2}{\gamma} \quad \ldots . (2).$$

Now the expression $\mathrm{X}\frac{dy}{ds} - \mathrm{Y}\frac{dx}{ds}$ is the result obtained by resolving the forces X and Y in the direction of the normal, and which, therefore, if the body were at rest, would, when taken negatively, denote the resistance of the curve; but being in motion, the curve suffers an additional resistance expressed by $\frac{v^2}{\gamma}$. We must here remark, however, that we have considered, in the above reasoning, the resistance R to be offered by the concave side of the curve; but if, on the contrary, the body press against the convex side, then R will not be the sum but the difference of the resistances, that is, the re-

sistance expressed by $\frac{v^2}{\gamma}$ must be subtracted from the resistance which the curve would oppose if the body were at rest.*

When the body is retained on the curve by the force of gravity only, then $Y=0$ and $X=-g$; therefore, in this case,

$$R=g\frac{dy}{ds}+\frac{v^2}{\gamma} \quad \ldots (3.)$$

As the pressure $\mp\frac{v^2}{\gamma}$, at any proposed point, depends solely upon the velocity at that point, it would remain the same if this velocity were produced by a primitive impulse only, and the motion to be uninfluenced by any acting forces, as indeed is plain from the expression $R=\pm\frac{v^2}{\gamma}$, for the resistance, when $X=0$ and $Y=0$. It is readily seen, therefore, that the pressure of which we speak arises entirely from the inertia of the moving body, or its tendency to move when at any point of the curve in the direction of a tangent and with its acquired velocity; this tendency necessarily causes it to exert a pressure against the deflecting curve, and which, as we have just seen, requires the curve to oppose the resistance $\pm\frac{v^2}{\gamma}$ in addition to the resistance necessary to oppose the normal effect of the acting forces.

A distinct name is given to the normal force or pressure $\mp\frac{v^2}{\gamma}$,* whose action on the body thus tends to repel it from the centre of curvature at that point of its path where its velocity is v; it is called the *centrifugal force.*

If the curve on which the body moves is a circle, and if we conceive that at the instant the velocity is v an attractive force expressed by $\frac{v^2}{r}$ be placed at the centre, then, if at the same instant all the other forces were destroyed, the body would continue to move in the same circle and with the same velocity v; for the repelling force tending to increase the distance of the body from the centre, is just balanced by the attractive force tending to confine the body to the curve, and moreover as v, if the body were left to itself, would continue unchanged throughout the curve (p. 151), the force which

* The student will not fail to remark that the upper sign applies when we consider the body to be moving on the *convexity* of the curve, in which case the pressure due to the acting forces is obviously diminished by this; and the lower sign applies when the body moves on the concavity, in which the pressure is necessarily increased by the same quantity.

counteracts the pressure arising from v at one point of the circular path will be competent to do so at every point; as, therefore, all pressure on the curve is destroyed, the motion of the body cannot be affected if the rigid curve were removed and the body left unconstrained.

This fact is at once deducible from the expression (2) for R, and indeed in a more general form; for in order that R may be 0, which is the same as saying in order that the motion may continue unchanged though the resisting curve be removed, we see that the normal effect of the applied forces must be equal and opposite to the force $\frac{v^2}{\gamma}$: so that if a force always expressed by this were always to act at the centre of curvature, corresponding to the radius γ, the rigid curve, be it what it may, might be removed without changing the trajectory. By this arrangement it is plain that the variable force $\frac{v^2}{\gamma}$ must itself move so as to describe the evolute of the curve described by the body; the evolute of a circle is a point, viz. the centre.

As the central force, or as it is usually called the *centripetal* force, necessary to retain a body in a circle is $F = \frac{v^2}{\gamma}$, and since, if t'' be the time of one revolution, we must have, in consequence of the uniform velocity, $v = \frac{2\pi\gamma}{t}$ $\therefore F = \frac{4\pi^2\gamma}{t^2}$, which expresses alike the intensity of either the centripetal or the centrifugal force.

In like manner, for any other circle of radius γ_1, and time t_1'', the centripetal or centrifugal force is

$$F_1 = \frac{4\pi^2\gamma_1}{t_1^2} \therefore F : F_1 :: \frac{\gamma}{t^2} : \frac{\gamma_1}{t_1^2};$$

hence, 1st. When the circles are equal, the centripetal or centrifugal forces are inversely as the squares of the times; and 2d. When the times of revolution are equal, the forces are as the radii of the respective circles.

Let us apply these results to an interesting problem, viz. to the determination of the centrifugal force at different places on the earth's surface, from knowing the time of one rotation on its axis.

The earth, by means of its diurnal motion, carries round with it, with a uniform velocity, every point on its surface in 86164 seconds. At the equator the radius γ is 20921185 feet, therefore the centrifugal force at the equator is

$$F=\frac{4\pi^2\gamma}{t^2}=\frac{4\pi^2 20921185}{86164^2}=\cdot 1112447 \text{ feet.}$$

As this force opposes the force of gravity, it follows, that if it did not exist, that is, if the earth did not revolve on its axis, the force of gravity instead of being what it really is, viz. $g=32\cdot 08818$ at the equator, would be $G=g+\cdot 1112447$, and thus the weight of any body would be a $\frac{\cdot 1112447}{32\cdot 08818}$ part more than it really is.

The ratio of G to F being

$$32\cdot 1994247 : \cdot 1112447 \text{ or } 289 : 1 \text{ nearly,}$$

$$\therefore F=\frac{G}{289}\ldots.(1).$$

Now every parallel to the equator being carried round in the same time t'' as the equator, we have, by representing the centrifugal force in the parallel whose latitude is l and radius γ, by F_1,

$$\gamma : \gamma_1 :: F : F_1$$

$$\therefore F_1=F\frac{\gamma_1}{\gamma}=\frac{G}{289}\cos. l\,.\,(2),$$

because it is evident that $\gamma_1=\gamma \cos. l$.

The force G of gravity is not diminished by the whole of the centrifugal force F_1, except at the equator, because this force acts in any parallel PAP′ (fig. 100,) not in the direction Pp opposite to gravity, but in the direction Pr, if, therefore, we decompose the force Pr in the perpendicular directions Pp, Pq, both, as well as Pr, being in the plane of P's meridian, we shall have, for the force Pp opposing gravity,

$$Pp=Pr\cos. pPr=F_1\cos. POQ=F_1\cos. l;$$

hence the expression for the diminution of gravity is (equa. 2,)

$$\frac{G}{289}\cos.^2 l;$$

which therefore varies as the square of the cosine of the latitude. The other force Pq, being tangential, tends to draw the particles of the revolving body from the poles towards the equator, and to cause it to assume the figure of an oblate spheroid; the expression for this force is

$$Pq=F_1\sin. l=\frac{G}{289}\sin. l\cos. l=\frac{G}{578}\sin. 2\,l,$$

which therefore varies as the sine of twice the latitude.

From the foregoing principles, let it now be required to determine the time in which the earth must perform its diurnal revolution in order that the centrifugal force at the equator may be exactly equal to the force of gravity, or in order that a body may have no weight there.

Let F_1 represent the time of rotation, corresponding to which the centrifugal force is G, then, as the centrifugal forces* in the same circle are inversely as the squares of the times, we have, (equa. 1,)

$$t^2_1 : t^2 :: \frac{G}{189} : G \therefore Gt^2_1 = \frac{Gt^2}{189} \therefore t_1 = \frac{t}{\sqrt{189}} = \frac{t}{17}.$$

Hence if the diurnal rotation of the earth were performed in a 17th part of the time really occupied, or if it were to turn round 17 times as rapidly, bodies at the equator would lose all their weight, and would, therefore, if placed at a small distance above the surface, remain suspended without any visible support; if the earth were to revolve still more rapidly than this, no body could remain on its surface: every thing would be repelled from it by the centrifugal force. This inference it must be remembered is, as well as that implied in equation (2), on the supposition that the earth retains its spherical figure during its rotation, which however is not strictly correct on account of the oblique influence of the centrifugal force tending to elevate the equator and to depress the poles, thus giving to the earth a spheroidal form. It is demonstrated that a fluid spheroid of the same density as that of the earth, cannot remain in equilibrium if it revolve in a shorter time than $2^{h.}\ 25'\ 26''$.*

(126.) The general equations (2) and (3), at page 151, contain all that is necessary for the determination of the motion of a body down any given curve by the action of gravity; or indeed of any force g acting in parallel lines. The theory of the pendulum, a highly important subject, is established by their aid, and to this theory we shall apply them in the following chapter. We may remark, however, before entering upon this, that the expression (1) for the velocity at page 150, is general for all hypothesis of the acting forces, and when this velocity is determined and the curve given, the time will be given by the equation $t = \int \frac{ds}{v}$, in the particular case, however, where the body is acted upon by a single centre of force, varying according to some function of its distance, then, for the determination of the velocity, it will be proper to use the expression (1) at page 147, taking the integral with a negative sign if the force be attractive, tending to diminish the co-ordinates, and taking it with a positive sign if repulsive; the axes of reference too must here originate at the centre of force.

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* See *Professor Airy's Tracts*, page 150, second edition; or the *Théorie Analytique du Système du Monde*, of *Pontécoulant;* tom. ii., page 400.

CHAPTER III.

ON THE SIMPLE PENDULUM.

(127.) A SIMPLE pendulum is considered to be a material point, attached to a thread or rod without weight, and oscillating about a fixed axis connected with the other extremity of the rod. Such a pendulum, it is evident, can have no physical existence, yet it is convenient to discuss the theory of such an imaginary pendulum, because, as will be shown in a subsequent chapter, whatever be the oscillating body, there may always be found a point at which, if a single particle were placed and connected by a rod, without weight, to the point of suspension, the oscillations of this simple pendulum would be performed in the same time as those of the compound body; and all the circumstances of its angular motion would be the same, and thus any pendulum may be reduced to an equivalent simple pendulum.

The moving point which we here consider, is confined to the curve in which it moves by the thread, the accelerating force being gravity; hence, the tension suffered by the string at any point of the path, must be equivalent to the pressure which would be sustained by the curve at that point if it were rigid, and the moving point were unconnected with the thread. The constraining forces being equivalent, the theory developed in the preceding chapter becomes immediately applicable to the motions of simple pendulums; these motions, although usually in circular arcs, may nevertheless be in any curves whatever, for the thread as it oscillates to and fro may be forced to wrap itself about curves springing from the point of suspension, and thus the material point will be forced to describe curves which are the involutes of these; and in this way we may have *circular pendulums*, *cycloidal pendulums*, &c. Of these, however, the circular pendulum is the most simple and important.

(128.) It will contribute to the convenience of the student to bring together in this place those formulas distributed in the preceding chapter, which are required in the problems we are about to give; these formulas are as follow:

$$\text{velocity} = v = \sqrt{2g(h-x)} \cdots\cdots (A),$$

$$\text{time} = t = \int \frac{ds^*}{\sqrt{2g(h-x)}} \cdots\cdots (B),$$

* If we consider the arc s in this expression to be the arc of descent, this, being measured from the origin or lowest point, must diminish as the time in-

$$\text{centrifugal force} = f = \frac{v^2}{\gamma} = \frac{2g(h-x)}{\gamma} \ldots\ldots (\text{C}),$$

$\text{tension} = \text{T} = g\frac{dy}{ds} + \frac{2g(h-x)}{\gamma}$. (D), or, since $\frac{dy}{ds}$ expresses the cosine of the angle which the normal makes with the axis of x, if we call this angle α we may write the last expression thus:

$$\text{tension} = g\cos.\alpha + \frac{2g(h-x)}{\gamma} \ldots\ldots (\text{E}),$$

which form will be sometimes most convenient to use, (see prob. IV. following.)

PROBLEM I. (129.) To determine the time of oscillation in a circular pendulum (fig. 101).

Taking the origin of the vertical and horizontal axes at the lowest point of the curve, and calling the radius of the circle or the length of the rod r, we have, for the equation of the path,

$$y^2 = 2rx - x^2 \;\therefore\; \frac{dy^2}{dx^2} = \frac{(r-x)^2}{2rx - x^2}$$

$$\therefore\; ds = \sqrt{\{1 + \frac{dy^2}{dx^2}\}}\,dx = \frac{rdx}{\sqrt{\{2rx - x^2\}}}$$

consequently, by equation (B) above, the expression for the time is

$$t = \int \frac{ds}{\sqrt{\{2g(h-x)\}}} = \frac{r}{\sqrt{\{2g\}}} \int \frac{dx}{\sqrt{\{(h-x)(2rx-x^2)\}}}.$$

The differential expression under the integral sign may be put under the more convenient form $\frac{dx}{\sqrt{\{2r\}}\sqrt{\{hx-x^2\}}}(1-\frac{x}{2r})^{-\frac{1}{2}}$; which shows, that if the second factor be developed by the binomial theorem, the differential in question will be reduced to a series of others all of the form $\frac{x^m\,dx}{\sqrt{\{hx-x^2\}}}$, which we know to be an integrable form.

These details we have entered into at length in the *Integral Calculus*, page 93, and the result is, that the proposed integral, taken between the necessary limits, that is from $x=0$, the lowest point of the curve, to $x=h$, the point of departure, is

creases, and therefore ds in this formula is negative; but if we refer to the arc of ascent, the time and arc increase together, and ds is positive, we shall therefore always consider the formula as applied to the ascending arc, since the time will be the same whether the body descend from the height x to the lowest point, or ascend from the lowest point to the height x; or whether it descend from h to x, or ascend from x to h. On this hypothesis, therefore, the integral (B) commences at $x=x$ and ends at $x=h$; for the descending arc, on the contrary, the integral commences at $x=h$ and ends at $h=x$.

$$\int_0^h \frac{dx}{\sqrt{\{(h-x)(2rx-x^2)\}}} =$$
$$\frac{1}{\sqrt{\{2r\}}}\{1+(\frac{1}{2})^2\frac{h}{2r}+(\frac{1\,.\,3}{2\,.\,4})^2\frac{h^2}{2^2\,r^2}+(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6})^2\frac{h^3}{2^3\,r^3}+\&c.\}\ \pi,$$

consequently for 2 t'', or the time of a complete oscillation, we have

$$2\,t=\pi\sqrt{\frac{r}{g}}\{1+(\frac{1}{2})^2\frac{h}{2r}+(\frac{1\,.\,3}{2\,.\,4})^2(\frac{h}{2r})^2+$$
$$(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6})^2(\frac{h}{2r})^3+\&c.\}\ \ldots.\ (1),$$

by means of which the time may be approximated to, to any degree of accuracy. The expression h is the versed sine of the arc of descent, or of half the whole path, and $\frac{h}{r}$ is the versed sine of a similar arc to radius 1, and therefore of the inclination of the rod to the vertical in its initial position; the smaller this inclination is the more convergent will the foregoing series be. Suppose, for instance, the initial inclination were 5°, then the versed sine of this being ·0038053, the second term of the series would be only

$$(\frac{1}{2})^2\frac{\cdot 0038053}{2}=\cdot 0004757,$$

and if the pendulum vibrated but one degree on each side of the vertical, then, since the versed sine of 1° is ·0001523, the second term of the series would be but $(\frac{1}{2})^2\frac{\cdot 0001523}{2}=\cdot 000019$, and, by supposing the arc of vibration less and less, the expression for the time of an oscillation would continually approximate to

$$2\,t=\pi\sqrt{\frac{r}{g}}\ \ldots.(2),$$

which will differ insensibly from the true time when the arcs of vibration are very small, that is, not exceeding about 4°; and therefore, for all arcs between this and 0, the times of vibration of the same pendulum will not perceptibly differ, that is, *in very small arcs the oscillations may be regarded as isochronal*, or as all performed in the same time.

Since the time occupied by a body falling freely through the height $\frac{1}{2}\,r$ is expressed by $\sqrt{\frac{r}{g}}$, it follows, from the foregoing expression, that in pendulums of such limited ranges or *amplitudes* as we have supposed, *the time of vibration is to the time of falling freely through half the length of the rod as* 3·14159 *to* 1.

It is an important matter to know exactly the length of a pendulum which will vibrate seconds, and combined with experiment the the general expression (1) will enable us to determine this length with perfect accuracy for any given arc of vibration.

Thus, let r, r' be the lengths of two pendulums vibrating in arcs of the same number of degrees, then, since $\frac{h}{r} = \frac{h'}{r'}$ the series within the brackets will be the same for each pendulum; hence, the times of oscillation of these pendulums will be to each other as

$$\pi\sqrt{\frac{r}{g}} \text{ to } \pi\sqrt{\frac{r'}{g}} \text{ or as } \sqrt{r} \text{ to } \sqrt{r'}:$$

the times of oscillation are therefore as the square roots of the lengths. Let the pendulum r make n oscillations in the same time t'' that the pendulum r' performs n' oscillations, then the respective times of a single oscillation will be $\frac{t''}{n}$ and $\frac{t''}{n'}$, which are to each other as $\frac{1}{n}$ to $\frac{1}{n'}$; hence, by the proportion just deduced,

$$r : r' :: \frac{1}{n^2} : \frac{1}{n'^2} :: n'^2 : n^2,$$

that is, *the lengths of pendulums vibrating in similar arcs are to each other inversely as the squares of the number of oscillations made by them in the same time.*

Now a seconds' pendulum must vibrate t times in t'', if, therefore, we take a pendulum of any length r, and count the number n of vibrations it makes in any time t'', we shall find the exact length r' of the seconds' pendulum vibrating in a similar arc by this proportion, viz.

$$t^2 : n^2 :: r : r' = \frac{n^2 r}{t^2} \quad \ldots . (3).$$

If the length of the seconds' pendulum be thus determined for very small arcs, we may thence, by help of the expression (2), determine the force of gravity at the place where the experiment is made for, as

$$1 = \pi\sqrt{\frac{r'}{g}} \;\therefore\; g = \pi^2 r' \cdot\cdot (4).$$

Now in the latitude of London r' has been found to be $= 39 . 14$ inches, consequently $g = \pi^2 \times 39 . 14$ in. $= 32{\cdot}19$ feet, the force of gravity in the latitude of London.

Knowing the length of the seconds' pendulum, it will be an easy matter, from the foregoing theorems, to find the time of vibration of a pendulum of any other length, or the length of a pendulum vibrating in any other time. Thus, the length of the seconds' pendulum being r', and that of any other r, we have by the first of those theorems, this expression for the number t of seconds, this last will vibrate in,

viz. $t = \sqrt{\frac{r}{r'}}$, $\therefore r = r'\,t^2$. Suppose, for example, we wanted to know the time of an oscillation of a pendulum 20 feet long, we should then have $t = \sqrt{\frac{240}{39\,.\,14}} = 2\,.\,5$ nearly, so that the time would be about 2 seconds and a half.

Again, if we wanted to know the length of a pendulum that should oscillate once in ten seconds, then we have

$$r=39\,.\,14\times 10^2=3914\,.\text{ inches.}$$

We may also readily determine the number of seconds lost or gained in a day by lengthening or shortening a seconds' pendulum by any proposed quantity; for, from the equation (3), we have

$$r = \frac{r'\,t^2}{n^2},$$

where r' is the length of the seconds' pendulum, and $t=86400$, the number of seconds in 24 hours, n being the number of times the altered pendulum r oscillates in 24 hours. Suppose $r=r'+p$, and $n=t-q$, p will then be the error in length of the altered pendulum, and q the consequent deficiency in the number of vibrations, or the loss in seconds, and we shall have

$$r'+p=\frac{r'\,t^2}{t^2-2\,tq+q^2}=r'+2r'\frac{q}{t}+\&c.$$

If the loss amount to but a few seconds, the powers of $\frac{q}{t}$ may obviously be neglected without sensible error, and we shall thus have

$$p=2r^1\frac{q}{t}\text{ and }q=\frac{pt}{2r'}$$

the first equation showing the increase of length corresponding to a given loss, and the second showing the loss consequent upon a given increase of length; and the expressions hold when the pendulum is diminished by p; q then expressing the gain.

Hitherto we have considered the pendulums compared to oscillate at the same place; but it is a very important inquiry to determine the lengths of pendulums oscillating seconds at different places on the earth's surface, as such a determination readily leads to the discovery of the true figure of the earth. If we represent by G and g the intensities of gravity at any two places, and by r and r' the lengths of the corresponding seconds' pendulums, then, by equation (4), we shall have $r=\frac{G}{\pi^2}$, $r'=\frac{g}{\pi^2}$, so that *the intensity of gravity at any places varies as the length of the seconds' pendulum at those places*. But it is manifest that the intensity of gravity must

depend upon the figure and constitution of the earth, and accordingly it is proved, by the writers on Physical Astronomy, that, considering the earth to be a homogeneous spheroid of equilibrium, the intensity of gravity must vary as the normal, so that, from the foregoing equations, the normal to the earth's surface at any place varies as the length of the seconds' pendulum at that place; and thus when the lengths of the pendulum for any two known latitudes are accurately ascertained by experiment, sufficient data will be furnished for determining the ratio of the earth's polar and equatorial diameters, or for finding the ellipticity or spherical *compression*, as it is called, and by which is meant the ratio of the difference of these two diameters to the greater. But we shall make the determination of this ratio from the proposed data a distinct problem.

PROBLEM II.—(130.) To determine the compression or ellipticity of the earth by means of seconds' pendulums.

Let a, b represent the equatorial and polar semi-diameters, e the eccentricity, and $c=\frac{a-b}{a}$, the compression; then

$$e^2=\frac{a^2-b^2}{a^2}=\frac{a-b}{a}\times\frac{a+b}{a}=c\,\{2-c\}.$$

Now c is itself but a small fraction, and the square of it is too small to be worth regarding in this inquiry, so that we may consider the compression to be expressed by $c=\frac{1}{2}e^2$.

Let λ, λ' be the latitudes at which the lengths of the seconds' pendulums are l, l', then, since *(Diff. Calc.* p. 134,) the expressions for the normals at these latitudes are

$$\mathrm{N}=\frac{b^2}{a}\cdot\frac{1}{(1-e^2\sin.^2\lambda)^{\frac{1}{2}}},\quad \mathrm{N}'=\frac{b^2}{a}\cdot\frac{1}{(1-e^2\sin.^2\lambda')^{\frac{1}{2}}},$$

we have $l:l'::(1-e^2\sin.^2\lambda)^{-\frac{1}{2}}:(1-e^2\sin.^2\lambda')^{-\frac{1}{2}}$, that is, by expanding the two last terms by the binomial theorem, and omitting the square and higher powers of $e^2\sin.^2\lambda$ on account of their excessive smallness,

$$\begin{aligned} l:l' &:: 1+\tfrac{1}{2}e^2\sin.^2\lambda : 1+\tfrac{1}{2}e^2\sin.^2\lambda' \\ &:: 1+c\sin.^2\lambda \quad : 1+c\sin.^2\lambda' \end{aligned}$$

$$\therefore c=\frac{l'-l}{l\sin.^2\lambda'-l'\sin.^2\lambda}=\frac{1-\frac{l}{l'}}{\frac{l}{l'}\sin.^2\lambda'-\sin.^2\lambda},$$

which expression would be the value of the compression, if, as we have supposed, the earth were of uniform density. Such, however, is not the case, yet the conclusion just obtained will enable us to deduce the true compression, whatever be the law of the

earth's density, by the aid of the following very remarkable proposition discovered by *Clairaut*, viz. "Whatever be the law of the earth's density, if the ellipticity of the surface be added to the ratio which the excess of the polar above the equatorial gravity bears to the equatorial gravity, their sum will be $\frac{5m}{2}$, m being the ratio of the centrifugal force at the equator to the equatorial gravity."* Now the ratio which the excess of the polar above the equatorial gravity bears to the equatorial gravity, is no other than the ellipticity c, as determined upon the hypothesis of homogeneity; for, calling the polar gravity G and the equatorial gravity g, and recollecting that these are as the normals, we have

$$G : g :: b : \frac{b^2}{a} \therefore \frac{G-g}{g} = \frac{a-b}{a} = c;$$

consequently, whatever be the law of the earth's density if we call the ellipticity or compression C, we have

$$C+c = \frac{5m}{2} = \frac{5}{2} \cdot \frac{1}{289} \text{ (page 109,) and therefore}$$

$$C = \frac{5}{578} - \frac{1 - \frac{l}{l'}}{\frac{l}{l'} \sin.^2 \lambda' - \sin.^2 \lambda}.$$

From experiments at Madras $\lambda = 13° . 4' . 9''$, $l = 39 \cdot 0234$
From experiments at Melville Island

$$\lambda' = 74° . 47' . 12'', \ l' = 39 \cdot 2070,$$

from which $c = \cdot 0053478$ and as $\frac{5}{578} = \cdot 0086505$, therefore

$$C = \cdot 0033027 = \frac{1}{302}.$$

From this value it appears that the equatorial diameter of the earth exceeds the polar by about the 300th part of its whole length; that is, these diameters are to each other as 300 to 299, and this ratio agrees almost exactly with the ratio as determined by means of the actual measurement of degrees. See *Dr. Gregory's Trigonometry*, page 231; and *Airy's Tracts*, page 186.

PROBLEM III.—(131.) To determine the time of oscillation of a cycloidal pendulum (fig. 102).

When the axes originate at the extremity of the base of the

* See Professor Airy's Tracts, p. 174.

cycloid we have found for the length of any arc s (*Int. Calc.* p. 115,)*

$$s=4r-2\sqrt{\{2r(2r-x)\}} \ldots . (1);$$

but if we measure s from the vertex, then, since the length of the semicycloid is $4r$, we shall have, by subtracting the foregoing expression from this, and then removing the origin to the vertex, that is, substituting $-x+2r$ for x, we have, in the inverted cycloid (fig. 103,)

$$s=2\sqrt{2rx} \therefore ds=\sqrt{\frac{2r}{x}}dx \therefore t=\int\frac{ds}{\sqrt{2g(h-x)}}$$

$$=\sqrt{\frac{r}{g}}\int\frac{dx}{\sqrt{hx-x^2}}=\sqrt{\frac{r}{g}}\text{ versin.}^{-1}\frac{2}{h}x+C;$$

hence, from $x=x$ to $x=h$, $t=\sqrt{\frac{r}{g}}\{\pi-\text{versin.}^{-1}\frac{2}{h}x\}$, which expresses the number of seconds in descending from the altitude h to the altitude x; therefore the time of descent to the lowest point A at which $x=0$ is $t=\pi\sqrt{\frac{r}{g}} \ldots . (2)$.

This expression being independent of h, is very remarkable, inasmuch as it proves that *the time of descent to the lowest point is always the same from whatever point in the curve the body begins to descend.* The oscillations in a cycloid are, therefore, always isochronal.

The cycloidal pendulum must oscillate between the two equal cycloidal cheeks SB, SC, about which the thread SP wraps itself; the length of the pendulum being equal to that of the curve SB, or which is the same, of the semi-cycloid BA, and this from equation (1), is by putting $y=2r$, $s=4r$, so that calling the length of the pendulum l the expression for a complete vibration is, from equation (2),

$$2t=\pi\sqrt{\frac{l}{g}};$$

which is the same as the expression given in last problem for the time of vibration of a pendulum of the same length, in a very small circular arc.

PROBLEM IV.—(132.) When a body vibrates in a circular arc, to determine the tension of the string at any point (fig. 101).

* An obvious error has crept into the formula here referred to; instead of

$$s=\sqrt{2r}\int\frac{dy}{\sqrt{y}}=2\sqrt{2ry}+C$$

it should have been

$$s=\sqrt{2r}\int\frac{dy}{\sqrt{2r-y}}=-2\sqrt{2r(2r-y)}+C,$$

and C is determined from the condition that $s=0$ when $y=0$.

Here the cosine of the angle α, or PXA, which the normal makes with the axis of x, is obviously $\frac{r-x}{r}$. Hence, by the formula for the tension, we have

$$\text{tension} = g\frac{r-x}{r} + \frac{2g(h-x)}{r} = g\frac{r+2h-3x}{r}$$

at the lowest point, or where $x=0$ tension $=g\frac{r+2h}{r}$; which, if the body fall from $h=r$, becomes $3g$; that is, the body, when it comes to A, is acted upon by three times as much force as it would be if at rest there, and, therefore, the body stretches the string with three times its weight. The tension is, obviously, greatest at this point.

In order to determine the point at which the tension is 0, we have $g\frac{r+2h-3x}{r} = 0 \therefore x = \frac{r+2h}{3}$, the abscissa of the point at which the pendulum, let fall from the height h, produces no strain on the point of suspension.

To determine the point at which the strain is the same as when the body hangs at rest, we must equate the expression for the tension with g, we thus find $x=\frac{2}{3}h$.

PROBLEM V.—(133.) To determine centrifugal force and the tension of the string in the cycloidal pendulum.

We have already seen (prob. II.) that in the cycloid the expression for sin. α, or $\frac{dx}{ds}$, is $\frac{dx}{ds} = \sqrt{\frac{x}{2r}}$

$$\therefore \cos.\, \alpha \text{ or } \frac{dy}{ds} = \sqrt{\{1-\sin.^2\}\,\alpha} = \sqrt{\frac{2r-x}{2r}} \therefore \frac{dy}{dx} = \sqrt{\frac{\{2r-x\}}{2r}}$$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{2r-x}{x}}$$

By means of these expressions we may find the radius of curvature at any point (x, y,) from the formula, *Diff. Calc.* p. 124–5. It is

$$\gamma = -\frac{ds^3}{dx^3} \div \frac{d^2y}{dx^2} = -\left(\frac{2r}{x}\right)^{\frac{3}{2}} \div \frac{-r}{x\sqrt{\{x(2r-x)\}}}$$

$$= 2\sqrt{\{2r(2r-x)\}};$$

hence the expression for the centrifugal force is

$$f = \frac{v^2}{\gamma} = \frac{g(h-x)}{\sqrt{\{2r(2r-x)\}}};$$

and, for the tension, we have

$$T = g\sqrt{\frac{\{2r-x\}}{2r}} + \frac{g(h-x)}{\sqrt{\{2r(2r-x)\}}}.$$

At the lowest point, or where $x=0$, both the centrifugal force and tension are obviously greatest; the expressions for them being,

$$f=\frac{gh}{2r},\ T=g+\frac{gh}{2r}.$$

PROBLEM VI.—(134.) To determine the time of gyration of a conical pendulum.

When the pendulum, instead of vibrating in a vertical plane, is made to pass over, or generate, a conical surface, as in fig. 104, it is called a *conical pendulum.*

The motion of such a pendulum is due to three forces, viz. the tension t, of the string AC; the force of gravity g, in the direction AD, and the centrifugal force f, in the direction AB; and these three forces keep the body, A, at the same constant distance r from S; hence, resolving the tension t in the directions AS, AE, we have

$$t \cos. a=f=\frac{v^2}{r}(\text{art. } 126),\ t \sin. a=g;\ \text{therefore,}$$

$$\text{putting CS}=a;\ \frac{\sin. a}{\cos. a}=\frac{gr}{v^2} \text{ or } \frac{a}{r}=\frac{gr}{v^2}\ \therefore v^2=\frac{gr^2}{a},$$

but t'' being the whole time of gyration, we have

$$v^2=\frac{4\pi^2 r^2}{t^2}=\frac{gr^2}{a}\ \therefore\ t=2\pi\sqrt{\frac{a}{g}};$$

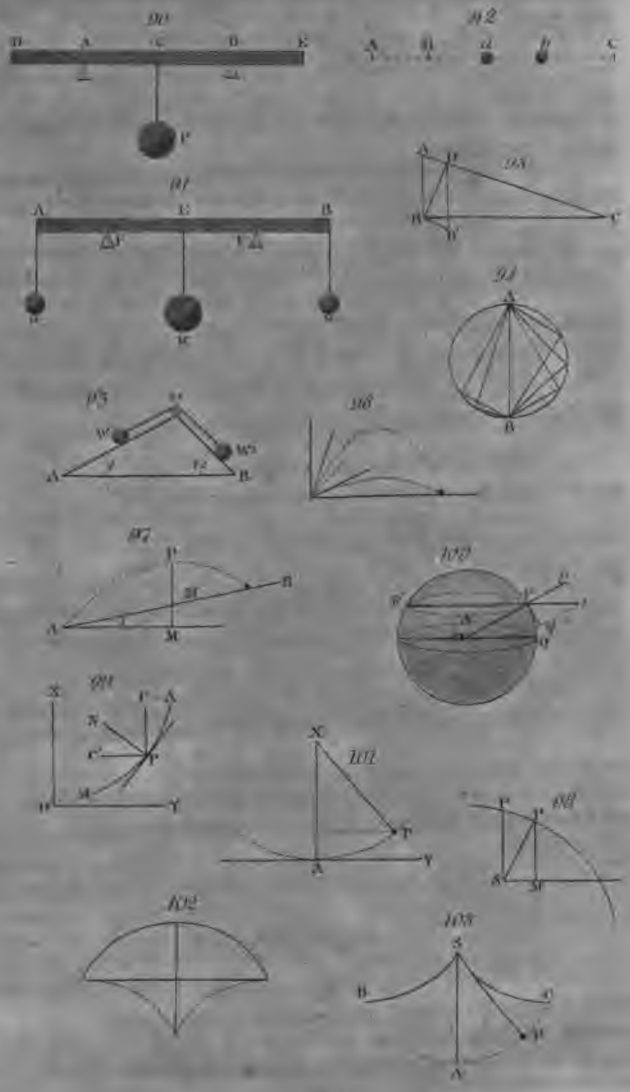
which expression, being independent of r, shows that the periodic time varies as the square root of the altitude of the conical surface described, whatever be the length of the pendulum, or the radius of the base of the cone.

We have seen (prob. I.) that the time in which a pendulum of length a vibrates in a small circular arc is expressed by $t=\pi\sqrt{\frac{a}{g}}$; hence the time of gyration of a conical pendulum is exactly double the time in which a simple pendulum, whose length is the height of the cone, would vibrate in a small circular arc.

CHAPTER IV.

ON CENTRAL FORCES.

(135.) THE motions of bodies, acted on by central forces, is a branch of the general theory, of so much importance, in the system of the world, that it will be proper to give it a distinct considera-



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tion, and to present the equations of motion with no more generality than may be requisite, in order to comprise the theory of a body's motion, when urged by a single centre of force.

It will be convenient here to put aside the use of rectangular co-ordinates, and to employ polar co-ordinates originating at the centre of force, so that P (fig. 105,) being the place of the body at any time t'', and S the centre of force, the point P will be determined from knowing $SP=r$, and the angle $PSX=\omega$.

The law of force, supposed to be some function of r, will be known when the general relation between r and ω, independently of t'', is known, and, conversely, this general relation, or the equation of the orbit, will be known, when the law of force is known; this we shall now proceed to show.

Let us suppose the force R to be attractive, then we know (121) that the velocity $\frac{ds}{dt}$, in the orbit, will be $\frac{ds^2}{dt^2}=-2\int R\,dr$ . . . (1).

Now. whatever be the independent variable, it is shown (*Int. Calc.* p. 118) that $(ds)^2=r^2(d\omega)^2+(dr)^2$; let then t be the independent variable, and we have, from (1),

$$r^2\frac{d\omega^2}{dt^2}+\frac{dr^2}{dt^2}=-2\int R\,dr \quad . \; . \; . \; . \; (2).$$

We moreover know (*Int. Calc.* p. 131) that the polar expression for the area generated in t'', viz. the area XSP is $\frac{1}{2}\int r^2 d\omega$, and we have seen (122) that this area is proportional to t, that is,

$$\int r^2 d\omega=ct \; \therefore \; r^2\frac{d\omega}{dt}=c \quad . \; . \; . \; . \; (3).$$

As ω is the angular space passed over, $\frac{d\omega}{dt}$ expresses the *angular velocity*, and the equation (3) shows that this angular velocity varies inversely as the square of the distance of the body from the centre of force. If we substitute in (2) the value of

$$\frac{d\omega}{dt}, \text{ in (3), we have } \frac{dr^2}{dt^2}+\frac{c^2}{r^2}=-2\int R\,dr=v^2 \quad . \; (4).$$

This equation, as it contains only the two variables r and t, which we see are at once separable, will enable us, when R is given, to find the general relation between r and t, and thus the distance of the body from the centre at any given time.

Again, by the same two equations, viz. (2) and (3), we may eliminate r and $\frac{dr}{dt}$, when $\int R dr$ is found, and we shall thus have a differential equation, involving only ω and t, from which the general relation between ω and t may be determined; and thus the position of the body at any time completely found. The two general ex-

pressions for t, thus obtained, will, when put equal to each other, obviously represent the path of the body. But this will be better done by eliminating at once dt* from the equations (2) and (3), by which we get for the differential equation of the orbit
$\frac{c^2}{r^2}\{\frac{1}{r^2}.\frac{dr^2}{d\omega^2}+1\}=-2\int \mathrm{R}\,dr=v^2$ .... (5); which will become somewhat simplified by putting u for $\frac{1}{r}$, as it then takes the form

$$\frac{du^2}{d\omega^2}+u^2=\frac{2}{c^2}\int\frac{\mathrm{R}}{u^2}\,du \text{ .... (6);}$$

or which is the same thing, $v^2=c^2\{\frac{du^2}{d\omega^2}+u^2\}$ .... (7.)

The first of these forms, after the integration $\int\frac{\mathrm{R}}{u^2}du$ has been performed, is the differential equation of the orbit, which, by integration, will become the algebraical equation of the curve. If, however, the orbit is already known, and we require to determine the law of force, the operation will be easier, as no integration will be necessary; thus, by differentiating (6), we have

$$\frac{d^2u}{d\omega^2}+u=\frac{\mathrm{R}}{c^2u^2}\therefore \mathrm{R}=c^2u^2\{\frac{d^2u}{d\omega^2}+u\} \text{ .... (8).}$$

These equations, or in fact the single equation (5), contains the whole theory of central forces, at least as far as regards the nature of the orbit, the law of the force, and the velocity of the body at any point. When the time enters into consideration the equations (2), (3), and (4) become useful.

(136.) Having established these equations, it would be easy now to deduce from them a variety of other forms, but we shall not detain the student by so doing. One or two transformations, however, deserve to be noticed, on account of their utility.

Looking to the terms within the brackets in equation (5), as connected with the angle P, (fig. 106,) we observe that they denote

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* We must caution the student against supposing that we here depart from the theory laid down in the Differential Calculus, by seeming to view dt as a finite quantity instead of absolutely nothing. It must be remembered that in every formula, containing only first differential coefficients, the independent variable may be always considered as entirely arbitrary; that is, every coefficient such as $\frac{d\omega}{dt}$, may always, without at all altering its value, be changed into the more general form $\frac{(d\omega)}{(dt)}$, (see Diff. Calc. p. 99.) In the above, therefore, we tacitly suppose this change to be effected, and eliminate, in fact, the finite quantity (dt).

(*Diff. Calc.* p. 119)* $\frac{1}{\tan.^2 \angle P}+1=\frac{1}{\sin.^2 \angle P}$.

Now, if from the pole a perpendicular p be demitted upon the tangent at P, its length will be $p=r \sin. \angle P$; hence we may transform the equation (5) into

$$\frac{c^2}{p^2}=-2\int R\, dr=v^2 \quad (9) \therefore R=\frac{c^2}{p^3}.\frac{dp}{dr}=\frac{v^2}{p}.\frac{dp}{dr} \quad (10);$$

an expression of remarkable simplicity.

(137.) Another useful and simple form is obtained by introducing the expression for the chord of the osculating circle drawn from P through the pole. This expression is thus deduced: from the centre of curvature C (fig. 107,) draw CM perpendicular to the radius vector, then PM will, obviously, be half the chord of curvature; upon the tangent PT demit the perpendicular ST$=p$: then the angle TPM being equal to the angle PCM the triangles STP, PMC, are similar, therefore,

$$\text{chord}=2\,\text{PM}=2\frac{\text{PC . ST}}{\text{SP}}=2\gamma\frac{p}{r} \;\ldots\ldots (1).$$

Now if we join SC$=a$, and draw the perpendicular ST$'$=PT, then, of whatever curve S is the focus and CP$=\gamma$ the radius of curvature, we always have, (*Geom.* p. 35,)

$$a^2=r^2+\gamma^2-2\gamma p \;\therefore\; 0=2r\frac{dr}{dp}-2\gamma \;\therefore\; 2\gamma=2r\frac{dr}{dp};$$

and, substituting this in (1), we have chord$=2p\frac{dr}{dp}$; and, consequently, the equation (10), art. (136), becomes by substitution $R=\frac{2v^2}{\text{chord}}$. Moreover, since from (10), $v^2=Rp\frac{dr}{dp}=R\times\frac{1}{2}$ chord; and since we likewise know, if the force R were to remain constant and to draw a body let fall from P along the chord of curvature, that when it should have fallen through $\frac{1}{4}$ the chord we should have $v^2=R\times\frac{1}{2}$ chord, we infer that *the velocity in the curve*, at any point, *is the very same as the body would acquire in falling through one fourth the chord of curvature, supposing the force, at that point, to remain constant.*

(138.) The force F, which placed at S, would compel the body at P to revolve round the centre, at the same distance SP, with the angular velocity, it actually has when at P, is called the centrifugal force at P, or rather the centrifugal force is equal and opposite to

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* At the top of the page here referred to, the expressions $r\frac{dr}{d\omega}$ and $-r^2$ should be interchanged

this. The expression for the angular velocity is, as we have already seen, $\frac{d\omega}{dt}$ which is no other than the actual velocity in the small circle which a point in the radius vector at the unit of distance from S describes as SP revolves;* the velocity, therefore, due to the force F, of which we speak, is $r\frac{d\omega}{dt}$; but when the body moves in a circle the centrifugal force is expressed by the square of the velocity divided by the radius, so that we have here

$$F=r\frac{d\omega^2}{dt^2} \therefore F=\frac{c^2}{r^3} \text{ (equa. 3, art. 135).}$$

If the force R, really placed at S, were but just sufficient to retain the body at P in a circle, either of these values of F would express its intensity; whatever additional influence, therefore, R actually exerts, or whatever influence short of this R exerts, it must be wholly employed in diminishing, or increasing the radius vector SP; this portion of force, therefore, is truly represented by $\pm\frac{d^2r}{dt^2}$; the upper sign applying when the radius vector increases, and the lower when it decreases; or omitting, as usual, the signs before the differential coefficients, we have, for the intensity of the central or *centripetal* force R,

$$R=r\frac{d\omega^2}{dt^2}-\frac{d^2r}{dt^2}=\frac{c^2}{r^3}-\frac{d^2r}{dt^2} \ldots\ldots (1).$$ The force $\frac{d^2r}{dt^2}$ is called the *paracentric force*, and the velocity $\frac{dr}{dt}$, due to it, the *paracentric velocity*. This velocity we may readily express for any point in terms of the coordinates, independently of t. Thus substituting the value of $-2\int Rdr$, in equation (5), art. (135) in the equation (4), which precedes it, and we have

$$\frac{dr^2}{dt^2}=\frac{c^2}{r^4}\cdot\frac{dr^2}{d\omega^2}=(\textit{paracentric velocity})^2.$$

The paracentric force is (equa. 1) the difference between the centripetal and centrifugal forces.

We shall now proceed to illustrate this theory.

On the Motions of the Planets.

(139.) Before *Newton's* discovery of the law of universal at-

* For as the angle ω is described in t'', the point to which we allude describes the arc $1 \times \omega$; hence the velocity, being the differential coefficient of the space, with respect to the time, is $\frac{d\omega}{dt}$.

traction the paths in which the planets revolve about the sun had been ascertained by observation; and the following laws, discovered by *Kepler*, and afterwards called *Kepler's laws*, were known to be true. They are the three following:

I. *The radius vector of every planet describes about the sun as a pole, equal areas in equal times.*

II. *The path of every planet is an ellipse, having the sun in one of its foci.*

III. *The squares of the times of revolution are as the cubes of the mean distances from the sun, or as the cubes of the major axes of the orbits.*

(140.) From these facts, revealed by observation, let us now deduce the law of attractive force, on which they must necessarily depend. In order that nothing may be assumed in this inquiry, let us, before we bring into application the preceding theory, show that this force, whatever it be, must be directed towards the sun. In order to this take the centre of the sun for the origin of the rectangular axes, these being in the plane of the orbit, then the components of the force acting on the body at any time t'', in directions parallel to these axes, will be

$\frac{d^2x}{dt^2}=X$, $\frac{d^2y}{dt^2}=Y$, from which we get, as at art. (122),

$$\frac{d.(ydx-xdy)}{dt^2}=Xy-Yx \ldots . (1).$$

We have, moreover, seen that $\int ydx - \int xdy$ is the double area described by the radius vector about the sun, during the time t'' (122), and since by Kepler's first law, this area is always proportional to the time, we may generally represent it by ct, c being constant; hence, we have, by differentiation, $\frac{ydx-xdy}{dt}=c$; and, differentiating again, we have $\frac{d.(ydx-xdy)}{dt^2}=0 . (2)$; consequently, (1) $Xy-Yx=0 \therefore \frac{Y}{X}=\frac{y}{x}$; that is, the lines represented by X, Y, are proportional to those represented by x, y, (fig. 108), and, therefore, PR is in the same line as PS, that is, the resultant of the forces on P is in the direction PS, so that the planet P moves under the influence of a central force at S, the place of the sun. We infer that this force must be attractive and not repulsive, because, from the second law, the curve PP′ is always concave to S, and, therefore, P is drawn from its wonted path Pp towards S.

(141.) Having established this, we have now only to compare the polar equation of the orbit with the equation (8), at page 168,

in order to determine the value of R the force of attraction on the planet. In the ellipse referred to the focus, the expression for r, is (*Anal. Geom.* p. 165,) $r=\frac{a(1-e^2)}{1+e\cos.\theta}$; in which a is the semi-major axis, OB, (fig. 110,) e the ratio of the eccentricity OS to the semi-major axis and θ the variable angle PSB. But, instead of SB, let us take any other fixed axis, SX, making with SB the angle BSX$=\alpha$, then calling PSX, ω, $\theta=\omega-\alpha$, and, therefore,

$$r=\frac{a(1-e^2)}{1+e\cos.(\omega-\alpha)}\therefore\frac{1}{r}=u=\frac{1+e\cos.(\omega-\alpha)}{a(1-e^2)}\ldots.(1).$$

Differentiating this, with respect to the variable angle ω, we have

$$\frac{du}{d\omega}=\frac{e\sin.(\omega-\alpha)}{a(1-e^2)};$$

and differentiating again, $\frac{d^2u}{d\omega^2}=-\frac{e\cos.(\omega-\alpha)}{a(1-e^2)}$. (2); adding this to equation (1) above, we have

$$\frac{d^2u}{d\omega^2}+u=\frac{1}{a(1-e^2)}.$$ Hence, by equation (8), page 168

$$R=\frac{c^2u^2}{a(1-e^2)}=\frac{c^2}{a(1-e^2)}\cdot\frac{1}{r^2}\ldots.(3)$$

The coefficient of $\frac{1}{r^2}$, in this expression for the central force is constant for the same orbit; hence *every planet is retained in its orbit by an attractive force residing in the sun, and varying in intensity inversely as the square of the distance at which it acts.*

(142.) For the velocity at any point, we have

$$v^2=-2\int Rdr=-\frac{2c^2}{a(1-e^2)}\int\frac{dr}{r^2}=\frac{2c^2}{a(1-e^2)}\cdot\frac{1}{r}+C;$$

to determine the constant C we must know, à priori, the velocity at some given distance; or we must know, at what distance and with what velocity the planet is originally projected into space. Calling this primitive velocity v_1, and the corresponding distance r_1, we have

$$v^2=v_1^2+\frac{2c^2}{a(1-e^2)}\left\{\frac{1}{r}-\frac{1}{r_1}\right\}\ldots.(1);$$

hence the velocity is greatest when r is least, that is, at that extremity of the major diameter which is nearest to the sun: this point is called the nearer *apse*, the curve being there perpendicular to the radius vector. At the opposite point, or farther apse, the velocity is least, since at this point r is greatest.

It may be remarked that the coefficient $\frac{c^2}{a(1-e^2)}$, which occurs in the foregoing expressions for R and v, is the value of the attracting force at the unit of distance from the centre, being what R becomes when $r=1$; $\frac{1}{2}c$ always represents the space described by the radius vector in one second of time.

(143.) The equation (9) at page 169, will furnish a simpler expression for the velocity than that just deduced; for the perpendicular p from the focus on the tangent being (*Anal. Geom.* p. 139.)

$$p^2=\frac{b^2r}{2a-r}=\frac{a^2(1-e^2)r}{2a-r}$$

$$\therefore v^2=\frac{c^2}{p^2}=\frac{c^2}{a^2(1-e^2)}\cdot\frac{2a-r}{r}\quad\ldots\ldots(2)$$

It will be easy to compare this velocity with that which a body would have revolving in a circle at the same distance r, and about the same centre of force; for, calling this velocity v', we know that $R=\frac{v'^2}{r}\therefore v'^2=Rr$; hence, referring to the value of R, in equation (3) above, we have

$$\text{vel.}^2\text{ in ellipse : vel.}^2\text{ in circle} :: \frac{2a-r}{ar}:\frac{1}{r}::2a-r:a;$$

that is, as the distance of P from the empty focus of the ellipse to the semi-major axis.

From the same expression for v^2 we learn that the velocity at the *mean distance*, $r=a$, that is, at the extremity of the minor axis, is a mean proportional between the velocities at the apsides or at the distances $r=r'$, and $r=2a-r'$.

(144.) In the expression for the force (3) the quantities a, e, c, which enter, are different for different planets; we cannot conclude, therefore, from this expression, whether, like terrestrial gravity, this force is independent of the magnitude and constitution of the body attracted, that is, whether at the same distance r, or at the unit of distance, R has, in all cases, the same value; but Kepler's third law, which we have not hitherto used, enables us to establish this point. Let T'' be the time of a complete revolution of any planet P; then, c being the double area described in 1'', cT will, by the first law, be the double area described during a whole revolution, that is, it will express twice the area of the ellipse; but the area of this ellipse is (*Int. Calc.* p. 123 art. 68) $\pi a^2\sqrt{\{1-e^2\}}$;

$$\therefore cT=2\pi a^2\sqrt{\{1-e^2\}}\therefore\frac{c^2}{a(1-e^2)}=\frac{4\pi^2a^3}{T^2}.$$

In like manner with respect to any other planet P'

$$\frac{c'^2}{a'(1-e'^2)}=\frac{4\pi^2 a'^3}{T'^2}$$

But, by Kepler's third law,

$$T : T' :: a^3 : a'^3 \therefore \frac{c^2}{a(1-e^2)}=\frac{c'^2}{a'(1-e'^2)};$$

and each of these expressions denotes the influence of the central force on each of the planets P, P′, at the unit of distance, these influences, therefore, being the same, the force is of a similar nature to that of terrestrial gravity, influencing all bodies alike at the same distance from the centre of force.

(145.) It may here be remarked that the same law of force (equa. 3) would be established if we did not know, from observation, that the paths of the planets were ellipses, but only that they were conic sections; for the equation (3) would remain the same except as relates to the value of e which is either equal to, less than, or greater than 1, according as the curve is a parabola, an ellipse, or a hyperbola. Hence *if a body move in a conic section, in virtue of a central force at the focus, it must vary inversely as the square of the distance at which it acts.*

(146.) Let us now proceed to the inverse problem, viz. to the determination of the orbit which a body must necessarily describe about a central force, which varies inversely as the square of the distance at which it acts; this being the law of force which, as we have before remarked, was discovered by *Newton* to be that which governs the planetary motions.

Let $R=\frac{h}{r^2}$, h being the intensity of the force at the unit of distance, or when $r=1$, then substituting this expression for R in the general equation (5), page 168, we have for this particular law of force the following expression for the velocity, viz.

$$v^2=\frac{c^2}{r^2}\left\{\frac{1}{r^2}\cdot\frac{dr^2}{d\omega^2}+1\right\}=\frac{2h}{r}+C\ldots(1);$$

the integral of which is the equation of the orbit.

But, by article 143, equation (2), when the orbit is a conic section

$$v^2=\frac{c^2}{r^2}\left\{\frac{1}{r^2}\cdot\frac{dr^2}{d\omega^2}+1\right\}=\frac{c^2}{a^2(1-e^2)}\cdot\frac{2a-r}{r}\ldots(2);$$

the integral of this equation is, therefore, the equation (1) p. 172, involving the arbitrary constant α, and this, be it observed, is true whatever be the values of the constants a and e. But if these be determined so that

$$(3)\ldots h=\frac{c^2}{a(1-e^2)},\ C=-\frac{h}{a}\ldots(4);$$

the two equations (1), (2) will, obviously, become identical; hence

with these conditions equation (1) will also be the integral of (1) above, and, in which e and a are fixed by the equations (3) and (4): (1), therefore, is the equation of the orbit sought.

We are warranted, therefore, in inferring the converse of the proposition at the close of last article, viz. that *if the central force vary inversely, as the square of the distance, the body must describe a conic section, having that force in its focus.*

(147.) Having thus determined the nature of the orbit, let us endeavour to ascertain its form, which will require the determination of the constants c and C, both of which depend upon the circumstances of the initial motion of the body.

C may be determined from knowing the initial velocity and distance, that is, the impulsion with which the body is launched into space, and the distance of the point of projection from the centre of force, call these respectively v_1 and r_1, then from equation (1)

$$C = v_1^2 - \frac{2h}{r_1} \dots (1).$$

To determine c requires that we know not only the point and velocity of projection, but also its direction; or the angle it makes with the radius vector at that point. Call this angle θ, then the initial velocity, in the direction perpendicular to the radius vector, is $v_1 \sin.\theta$, which is, therefore, equal to $r_1 \frac{d\omega}{dt}$; but equation (3), p. 167, $r_1^2 \frac{d\omega}{dt} = c$; consequently,

$$c = r_1 v_1 \sin.\theta \dots (2).$$

Hence the constants which enter into the equations (3) and (4) are determined in terms of the initial quantities; substituting them in those equations, we get, from (4), $a = \frac{-h r_1}{r_1 v_1^2 - 2h}$; and, from (3), $e = \sqrt{1 - \frac{c^2}{ah}} = \sqrt{1 + \frac{r_1^2 v_1^2 \sin.^2\theta}{h^2}(v_1^2 - \frac{2h}{r_1})}$.

By means of these two equations the orbit may be constructed; its form may be completely determined by the second equation alone, since the form depends entirely upon the value of e. It will be an ellipse if $e < 1$, an hyperbola if $e > 1$, and a parabola if $e = 1$; that is, the orbit will be an ellipse if $v_1^2 < \frac{2h}{r_1}$; an hyperbola if $v_1^2 > \frac{2h}{r_1}$;

$$\text{a parabola if } v_1^2 = \frac{2h}{r_1};$$

so that the same central force which governs the planetary motions, and causes them to describe ellipses, would have been equally competent to cause them to describe either hyperbolas or parabolas, but

not any other curves. For aught we know to the contrary, therefore, there may be planets governed by the same attractive influence, and moving in hyperbolic or parabolic orbits, and which, therefore, continually proceed onward in space without ever returning.

It is a singular fact, that, as the foregoing expression for a, the semi-major axis of the orbit, is independent of θ, the angle of projection, the major axis of the orbit, will be of the same length whatever be the angle of projection; the minor axis, however, will vary with this angle, seeing that its sine enters into the expression for the eccentricity.

As to the absolute lengths of these axes, they are at once determined when the initial conditions of the motion are given by means of the equations (3) and (4), since the constants which enter them are known in terms of these given conditions by equations (1) and (2).

The velocity of the body at any point of its orbit is given by the equation (2) art. (146), and, by substituting this expression, for v in the general equation (5), art. (135), we readily get a differential expression for the angle ω corresponding to that point, and, finally, the value of $d\omega$, given by this expression, substituted in the equation (3) of the same article, will furnish a differential equation between t and r, and thence the time of arriving at the proposed point becomes known.

In all these results the quantity h enters, which quantity denotes the value of the accelerative force at the unit of distance from the centre of attraction; or rather it expresses the number of these units in the linear space which measures the attractive force at the unit of distance. Now the attractive forces of the sun and planets, corresponding to any proposed distance, vary directly as their masses, therefore, whatever energy the unit of mass exerts at the unit of distance the mass M of the sun exerts M times as much, and the mass m of the planet, m times as much; the whole force, therefore, which the sun regarded as fixed, exerts on the planet at the unit of distance, is (art. 119) $\text{M}+m$ times as much as that exerted by the unit of mass at the unit of distance. Considering this latter to be the unit of force, and representing it by 1, accordingly $\text{M}+m$ will truly represent the attractive power exerted on the planet at the unit of distance, and, therefore, at r such units distant the expression for the attractive force will be $\frac{\text{M}+m}{r^2}$, which is the value of $\frac{h}{r^2}$ in the preceding formulas when we consider the action of the sun on a single planet only. We may also express the intensity of the solar and planetary attractions in terms of terrestrial gra-

vity; thus, calling the radius of the earth r_1, and m_1 its mass, we have, since the attractions are directly as the masses and inversely as the squares of the distances,

$$\frac{m_1}{r_1^2} : \frac{M}{r^2} :: g : \frac{Mr_1^2}{m_1 r^2} g,$$

the attractive force of the sun at any distance r; and, in like manner, the attractive force of the planet is $\frac{mr_1^2}{m_1 r^2} g$; hence their united influence is $\frac{M+m}{m_1} \frac{r_1^2}{r^2} . g$, the factor which multiplies g being an abstract number. This number will, of course, remain the same if we deliver the mass and distance each from its peculiar unit, regarding M, m, r^2, &c. to be numbers merely, in which case writing the expression thus $\frac{M+m}{r^2} . \frac{r_1^2}{m^1} g$; we see that $\frac{r_1^2}{m_1} g$ is that which we have taken above for the unit of attractive force; it is plainly the expression for the attraction of the mass 1 at the distance 1.

We here close the second section; for, to pursue these interesting inquiries further, we should be compelled to pass by matters more especially entitled to a place in a treatise on Mechanics. We have, however, given thus much of the first principles of Physical Astronomy, because we were unwilling altogether to omit touching upon a subject so highly calculated to excite the inquiries of the student, and because, moreover, we had indulged a hope of being able to unfold these first principles with more simplicity than is usually done.

SECTION III.

ON THE MOTION OF A SOLID BODY.

(148.) Hitherto we have limited our investigations almost solely to the motions of a single point: or, if we have at any time introduced the consideration of a moving body, it has usually been on the hypothesis, that the influence by which it moved was diffused uniformly through its mass, acting alike upon every particle; so that if any portion of the body were to be removed, all the influence, in virtue of which that particular portion moved, would be taken away too, and the remaining part of the mass would go on precisely in the same way as it would have done if accompanied by the part subtracted, and precisely as it would do if reduced to a sin-

gle particle. It is in this way that the forces of attraction are uniformly diffused through the masses of the attracted bodies, and to this description of forces our attention has been almost entirely directed hitherto. We have not, however, wholly overlooked those motions which result from pressure constantly pushing forward a mass of matter, not that we mean to mark any difference, as far as effects are concerned, between pressure dynamically considered, and an attractive force; for, as already observed at (104), the motion produced by an attractive force may be considered as due to the body's own pressure, as is manifest; but if it were to be urged by a pressure less than this, we ought obviously to expect a diminished acceleration; and if it were urged by a greater pressure, we should look for an increased acceleration; but, as just remarked, particular cases of this kind have already been considered, viz. in problems I. and II. at page 129; thus referring to the first of these problems, we find that the whole weight to which motion is given, is $W+W_1$, which represents the whole pressure of the mass moved; but the motion is due to a less pressure, viz. to $W_1 \sin. i_1 - W \sin.^s i$, and accordingly we find a diminished acceleration.

(149.) The pressure which thus moves a body is called a moving pressure, or rather a *moving force;* the acceleration due to such a force, or the force competent to produce any proposed acceleration, may be determined by the principle at page 121: thus, calling the whole pressure or weight of any mass M, W, the acceleration due to this pressure g, and that produced by any other pressure or moving force, F, we have $g : F :: W : W\frac{F}{g}$ = the moving force, which expression obviously represents the weight which the mass M would have if the accelerative force which impressed that weight were F.

The weights of bodies obviously vary with their mass, or with the quantity of matter they contain, and also, as the accelerative force which impresses weight; that is to say, weight varies conjointly as the mass and the accelerative force; we may, therefore, in these inquiries substitute for any weight W the quantity to which it is always proportional, viz. the quantity Mg; M being the mass or rather the number of times the body contains some fixed unit of mass, and g being the value of gravity, or the force which impresses weight on the mass. Representing the weight W by Mg, and putting ϕ for the moving force, we have, by the foregoing expression,

$$\phi = MF = M\frac{dv}{dt} = M\frac{d^2 s}{dt^2},$$

supposing, as we here do, that each particle of the moving mass has a common velocity at every instant.

As it immediately follows from this that $F=\frac{\phi}{M}$, we see that the acceleration is as the moving force or pressure directly, and as the mass inversely.

For the velocity at any time t'', we have, supposing ϕ constant, $dv=\frac{\phi}{M}dt \therefore v=\frac{\phi}{M}t$, so that the velocity acquired in any time t'', is the same, so long as the ratio, $\frac{\phi}{M}$, of the moving force to the mass moved is the same.

It is usual to call the product Mv of the moving mass by its velocity the *momentum*, so that as accelèrative force is represented by the differential coefficient of the velocity relatively to the independent variable t, the moving force is represented by the differential coefficient of the momentum relatively to t. Instead of mass, however, we may, if we please, always substitute weight, since, as remarked above, these are always proportional.

We shall terminate these introductory remarks by observing, that all the deductions at art. 105, respecting the acceleration, velocity, space, and time, apply equally here; and all the formulas employed there become suited to the present inquiry when $\frac{\phi}{M}$ is substituted in them for F; provided, as before mentioned, that the moving force impresses a common velocity on all the particles of the mass moved, so that the acceleration of the whole may be that of each particle.

CHAPTER I.

ON THE COLLISION OF BODIES.

(150.) In the present chapter we propose briefly to consider the circumstances of the motions of bodies moving in certain directions, from the effects of impulsion and impinging against each other.

Let us suppose that the quantity of matter which we assume for the unit is projected by any given impulsion; it will move with a constant velocity always proportional to the intensity of the force of impulsion (p. 118); this velocity, therefore, will always cor-

rectly represent the intensity of the force. If we take two such units, and two such impulsions act on them simultaneously, and in the same direction, the relative positions of the moving masses will be always preserved, so that if the two units were actually blended into one mass, and the two impulsions be thus made to coalesce and form a double impulsion, the same velocity as before would be impressed on the mass; and it is plain that in like manner if M such units be blended into one mass, which receives an impulsion of M times the intensity we have supposed applied to the unit, still the same velocity would be impressed; consequently, when any body whose mass is M moves from the effect of an impulsive force, the correct expression for the intensity of that force must be Mv, the mass into the velocity, that is the momentum, measures the impulsive force.

Knowing then how to estimate the intensity of impulsive force, we may proceed to consider the circumstances of

Direct Impact.

(151.) We shall first consider the bodies which impinge to be entirely inelastic, or of such a nature that they are blended by the impact into one mass, and our object will be from knowing the forces of the impinging bodies to determine the motion of the united mass.

Problem I.—Two inelastic bodies M, M_1 move in the same straight line with velocities v, v_1: to determine the velocity after impact.

The impulsive force on M is Mv; that on M_1 is $M_1 v_1$, and these two combined form the impulsive force which moves the blended mass $M+M_1$; but if V be the velocity of this mass, the impulsive force on it must be $(M+M_1)\,V$,

$$\therefore\ (M+M_1)\,V=Mv+M_1v_1\ \therefore\ V=\frac{Mv+M_1v_1}{M+M_1}\ ..\ (1)$$

the velocity required. If the body M_1 were moving in a direction opposite to M so as to meet it, then v_1 should be taken negatively, and in that case, $V=\dfrac{Mv-M_1v_1}{M+M_1}$. (2), and the sign of V thus determined will point out the direction in which the mass moves after impact.

If the body M_1 be at rest, then, since $v_1=0$, $V=\dfrac{Mv}{M+M_1}$. (3).

In all cases the momentum $(M+M_1)\,V$ after impact is the sum of the momenta before impact, those being taken with opposite signs which act in opposite directions; so that the momentum lost

by the one body by the collision, is precisely that which is gained by the other. In the last of the above cases (3), the momentum of the whole mass being $(M+M_1)V=Mv$, and the momentum gained by M, which was at rest being M_1V_1 this must be the momentum lost by M.

This loss of force in M shows that the mass M_1 opposes a resistance to the communication of motion, and M_1V expresses the value of that resistance, or the impulsive force necessary to balance it; this result is quite independent of the weight of the resisting body, seeing that we here consider only its mass; it is, therefore, the consequence of its inertia, which is therefore proportional to the mass.

(152.) Let us now consider the impact of two elastic bodies which, as in the former case, move so as to impinge at some point in the common line described by their centres of gravity.

Bodies of this class yield to the force of impact, and suffer a compression, and therefore a change of figure; and the elasticity is that inherent force which the body exerts to recover its original form. If the force thus exerted at every point throughout the whole depth of the impression, while the body is recovering its form, is equal to the impressing force at that point, the original form of the body must be perfectly restored, and the elasticity is then said to be perfect.* In this case, whatever velocity one body lost during the action of the force of compression, it afterwards lost just as much more during the action of the force of restitution; and whatever velocity the other body gained during the compression, it gained as much more during the restitution; for the compressing and restoring forces (which are no other than continued pressures, although operating for an exceedingly short time) are equal, and therefore equally oppose the motion of one of the bodies, and equally favour the motion of the other.

It is easily seen that the same would be true if only one of the bodies were perfectly elastic, and the other *perfectly hard*, or incapable of receiving an impression.

When the force of restitution is not equal to that of compression, but only the eth part of it, the elasticity is said to be imperfect, and e measures its relative intensity; it is competent to communicate only the eth part of the velocity due to perfect elasticity.

PROBLEM II.—Two elastic bodies M, M_1, moving with velocities v, v_1, strike with direct impact: to determine their velocities afterwards.

So long as the compression continues, the bodies move as one

* The restitution is moreover considered to occupy the same length of time as the compression.

mass, and therefore with the velocity $V=\frac{Mv+M_1v_1}{M+M_1}$, so that M will lose the velocity $v-V$ and M_1 will lose the velocity v_1-V, and these would express the velocities communicated, but in opposite directions, by the force of restitution, if the elasticity were perfect; but let it be only the eth part of perfect, e being a fraction, then the additional velocity lost by M will be $e\,(v-V)$, and by M_1, $e(v_1-V)$; hence the velocities after the impact will be

of the body M, $V'=v-(1+e)\,(v-V)\ \ldots.\ (1)$;

of the body M_1, $V''=v_1-(1+e)\,(v_1-V)\ \ldots.\ (2)$;

or, substituting for V its value above,

$$V'=v-(1+e)\frac{M_1\,(v-v_1)}{M+M_1}\ \ldots.\ (3),$$

$$V''=v_1-(1+e)\frac{M\,(v_1-v)}{M+M_1}\ \ldots.\ (4).$$

If $e=0$, that is, if the bodies are inelastic, then, as is plain from the equations (1) (2), the velocities after impact are $V'=V$, $V''=V$, as we know they ought to be. If $e=1$, that is, if the elasticity is perfect, then, from equations (3) and (4),

$$V'=v-2\frac{M_1\,(v-v_1)}{M+M_1}\ \ldots.\ (5)$$

$$V''=v_1-2\frac{M\,(v_1-v)}{M+M_1}\ \ldots.\ (6).$$

If we multiply each mass by the velocity of it after impact, we have, when the bodies are perfectly elastic,

$$MV'+M_1V''=Mv+M_1\,v_1\ \ldots.\ (7);$$

hence the sum of the momenta before impact is the same as the sum after impact.

If we subtract the equation (2) from (1), taking $e=1$, we have

$$V'-V''=v-2v-v_1+2v_1=v_1-v\ \ldots.\ (8);$$

hence the difference of the velocities is the same both before and after impact.

When the bodies are perfectly equal, as well as perfectly elastic, they exchange their velocities, each moving after the impact with the velocity the other had before the impact; this follows from putting $M=M_1$, in the equations (5), (6); if, therefore, one body be at rest, the other, which strikes it, will impart to it its entire velocity, and rest in its place.

From the equations (7) and (8) we have $M\,(V'-v)=M_1\,(v_1-V'')$; $V'+v=v_1+V''$; multiplying these together there results $M\,(V'^2-v^2)=M_1\,(v_1{}^2-V''^2)$ or

$$MV'^2+M_1\,V''^2=Mv^2+M_1v_1{}^2\ \ldots.\ (9);$$

that is, the sum of the products of each body into the square of its velocity is the same, both before and after impact. The mass into

the square of the velocity is called the *vis viva*, or *living force*; so that in the collision of perfectly elastic bodies there is no loss of vis viva.

If one of the bodies is an immoveable mass we may consider it in the light of a mass M_1 at rest, and infinitely large; on which supposition equation (3) gives for the velocity of M after impact $V' = -ev$ .... (10); that is, as we might expect, M would rebound with the same velocity with which it struck the immoveable mass. It is true that, conformably to our hypothesis, the immoveable mass is supposed to be perfectly elastic as well as the impinging body; but the motion would be the same if it were perfectly hard, because the force of restitution would still be the same as that of compression, and the whole of this would be exerted to repel the body.

On Oblique Impact.

(153.) When the centres of gravity of two impinging bodies do not move in the same straight line, yet if at the instant of collision, the shock which each receives be directed towards its centre of gravity, the effects will be calculable by the preceding formulas. Thus suppose two spherical bodies M, M_1, (fig. 111,) move so that their centres describe the lines PA, QB, and that they strike each other at the point C; we may decompose each of the velocities with which the bodies impinge into two, one in the direction of a common tangent to the bodies at C, and the other perpendicular to this tangent; the components perpèndicular to the tangent will be those to which the impact is entirely due, the other components will merely express the lateral velocity, or that parallel to the tangent plane, which the respective bodies had before impact, and which, as nothing opposes it, they must still retain. Hence, having determined the velocities consequent upon the direct impact due to the effective components of which we have spoken, by means of the formulas in the preceding problems, we shall then have to compound these velocities, each with what the body originally had in the direction parallel to the tangent plane at their point of concourse.

As an example, let us take the case where one of the spheres is at rest and infinite in magnitude, or, which is the same thing, let the body struck be an immoveable plane: let the velocity given to the impinging body be v, the angle its direction makes with the plane, that is, the *angle of incidence* α, then the components of the velocity are $v \cos. \alpha$, $v \sin. \alpha$; the latter of these being perpendicular to the plane produces the impact; therefore, if e measure the elasticity of the body, the velocity, after the impact, will be (equa. 10) $V' = ev \sin. \alpha$, and the direction will be perpendicular to the plane; hence the velocity, after impact, will be the resultant of the

velocities v cos. α, and $e\,v$ sin. α, of which the directions are perpendicular to each other; therefore the velocity of reflection will be $v\surd\{\cos.^2\alpha+e^2 \sin.^2\alpha\}$; and the angle α' of reflection will be

$$\tan.\ \alpha'=\frac{ev\ \sin.\ \alpha}{v\ \cos.\ \alpha}=e\ \tan.\ \alpha;$$

so that if the elasticity be perfect, the velocity and the angle of reflection will be respectively equal to the velocity and the angle of incidence.

For a more enlarged view of the theory of percussion and impact the student may consult *Dr. Gregory's Mechanics*, vol. i. and the *Mécanique* of *Poisson*, tom. ii.; and for a variety of examples reference may be made to *Bridge's Mechanics*, p. 150 et seq. We omit practical examples here, which, however, are very easily framed, in order to make room for more important matter.

CHAPTER II.

THE PRINCIPLE OF D'ALEMBERT.

(154.) Inquiries concerning moving forces may be sometimes considerably facilitated by the aid of a very simple and very general proposition, first introduced into Dynamics by *D'Alembert:* it may be regarded as a dynamical axiom, and may be announced as follows:

If there be any system of bodies A, B, C, &c. which in virtue of the forces applied to them would, if entirely free, receive the several velocities a, b, c, &c. but which, on account of their mutual connexion, receive instead the velocities α, β, γ, &c. then it is evident that if the velocity a, impressed on A were resolved into two, of which one is the velocity α, actually received, and the other some velocity α'; and if, in like manner, the velocity b impressed on B were resolved into two, viz. the actual velocity β, and some other β'; and if a similar decomposition be effected for each impressed velocity, the forces due to the component velocities α', β', γ', &c. if severally applied to the bodies A, B, C, &c. of the connected system would keep the system in equilibrium.

For by the decomposition, which has been effected, all the motion which actually has place must be due to the other components, and, therefore, those of which we speak must destroy themselves in consequence of mutual actions of A, B, C, &c. on each other.

It follows, therefore, that *there must always be an equilibrium between the impressed forces and the actual, or effective, forces,*

these latter being taken opposite to their real directions, that is to say, if to the body A we apply the force originally impressed, and also the force due to the actual motion of A, this latter being opposite to its real direction; and if we do the same with all the other bodies B, C, &c. the whole system will be kept in equilibrium. For although the effective force on either of the bodies is not equivalent to the impressed force, yet, as we have just seen, the impressed forces may be considered as resulting from the effective forces, combined with another set which destroy each other; hence as these equilibrate, the system must equilibrate by the combined action of the impressed forces with the effective forces taken in opposite directions.

The forces here spoken of are, of course, moving forces or simply momenta; that is, continued pressures, or pressures of but momentary duration.

As an illustration of the foregoing general principle we may take the problem already solved, at page 129, viz. to determine the motions of two weights W, W_1, along inclined planes, placed back to back, the weights being connected by a thread.

Let us first ascertain the impressed forces, or those in virtue of which the bodies would move if unconnected, these are evidently for W, W sin. i, and for W_1, W_1 sin. i. Let us now determine the effective or actual forces; for this purpose call the velocity of W, v; and that of W_1, v_1; then as we know (149) that the motive force is always equal to the *mass* multiplied by the acceleration, the effective forces are on W, $-\frac{W}{g}\cdot\frac{dv}{dt}$* and on W_1, $\frac{W^1}{g}\cdot\frac{dv_1}{dt}$. Now, by the foregoing principle, if these forces taken with contrary signs, that is, taken negatively, be simultaneously applied to the respective bodies with the former forces, the system will be in equilibrium; that is, there will be an equilibrium if to the two bodies W, W_1 at rest, there be applied the respective forces

$$W \sin. i. + \frac{W}{g}\cdot\frac{dv}{dt} \text{ and } W_1 \sin. i_1 - \frac{W_1}{g}.\frac{dv_1}{dt};$$

therefore, as these must pull the thread in opposite directions, we must have the equation

$$W \sin. i + \frac{W}{g}\cdot\frac{dv}{dt} = W_1 \sin. i_1 - \frac{W_1}{g}\cdot\frac{dv_1}{dt};$$

* Regard must, of course, be paid to the signs of the acting forces; so that if we consider a force applied to a body in one direction to be positive, we must consider any other force applied in an opposite direction to be negative; therefore, as we here suppose the effective force on W to pull it up the plane, we must take it negatively, because we have taken the impressed force on it, which would pull it *down* positively.

therefore, since v is necessarily equal to v_1, we have

$$\left\{\frac{W}{g}+\frac{W_1}{g}\right\}\frac{dv}{dt}=W_1 \sin. i_1 - W \sin. i$$

$$\therefore \frac{dv}{dt}=\frac{W_1 \sin. i_1 - W \sin. i}{W+W_1}g;$$

which is the value of the accelerative force.

This solution it may be observed is not so simple as that given upon different principles at page 129; but it may serve to illustrate D'Alembert's principle. It may not, however, be amiss to remark here that the solution of this and of other similar problems may be readily obtained by equating two different expressions for the effective moving force. Thus in the present problem the effective moving force is obviously $W_1 \sin. i_1 - W \sin. i$; it is also

$$\frac{W}{g}\cdot\frac{dv}{dt}+\frac{W_1}{g}\cdot\frac{dv_1}{dt},$$

and, therefore, we have the same equation as above.

We shall give another illustration of D'Alembert's principle in this place.

Two weights W_1, W, are attached, the one to a wheel, and the other to its axle, to determine their motions.

The impressed forces, or those in virtue of which the bodies would move, if free, are the weights themselves; the effective forces, or those in virtue of which they actually move, are as in the last problem

$-\frac{W}{g}\cdot\frac{dv}{dt}$ on W, supposing this to ascend, and $\frac{W_1}{g}\cdot\frac{dv}{dt}$ on W_1, hence the system will be in equilibrium when to the bodies W and W_1 there are applied the respective forces

$$W+\frac{W}{g}\cdot\frac{dv}{dt} \text{ and } W_1-\frac{W_1}{g}\cdot\frac{dv_1}{dt};$$

and as these acting at the extremities of the radii r and R, of the axle and of the wheel, tend to turn the system in opposite directions about the axis, we must have the equation

$$r\left(W+\frac{W}{g}\cdot\frac{dv}{dt}\right)=R\left(W_1-\frac{W_1}{g}\cdot\frac{dv_1}{dt}\right) \quad \ldots \quad (1);$$

the relation between v and v_1 is easily determined, for, as the wheel must turn in the same time as the axle, the velocities of the weights attached to them must be as their radii,

$$\therefore v_1=\frac{R}{r}v \;\therefore \frac{dv_1}{dt}=\frac{R}{r}\cdot\frac{dv}{dt} \quad \ldots \quad (2),$$

consequently the equation (1) is the same as

$$r\left(W+\frac{W}{g}\cdot\frac{dv}{dt}\right)=R\left(W_1-\frac{W_1}{g}\cdot\frac{R}{r}\cdot\frac{dv}{dt}\right)$$

$$\therefore \left\{ \frac{W_1}{g} \cdot \frac{R^2}{r} + \frac{Wr}{g} \right\} \frac{dv}{dt} = RW_1 - rW$$

$$\therefore \frac{dv}{dt} = \frac{R\,r\,W_1 - r^2\,W}{W_1\,R^2 + Wr^2}\,g,$$

which expresses the accelerative force on the ascending weight, and therefore for the velocity and space we have

$$v = \frac{RrW_1 - r^2\,W}{W_1\,R^2 + Wr}gt;\ s = \frac{R\,r\,W_1 - r^2\,W}{2\,(W_1\,R^2 + Wr)}gt^2.$$

For the acceleration of the descending weight we have, in virtue of the equation (2), $\frac{dv_1}{dt} = \frac{R^2\,W_1 - R\,r\,W}{W_1\,R^2 + Wr^2}\,g$

$$\therefore v_1 = \frac{R^2\,W_1 - R\,r\,W}{W_1\,R^2 + Wr^2}gt;\ s_1 = \frac{R^2\,W_1 - R\,r\,W}{2\,(W_1\,R^2 + Wr^2)}gt^2.$$

These two examples may suffice for the present to illustrate the application of D'Alembert's principle. We shall have frequent occasion to refer to it again in the course of the following chapters.

CHAPTER III.

ON THE MOMENTS OF INERTIA.

(155.) In treating of the rotation of a solid body, we shall always have to take account of that portion of the impressed forces which must necessarily be employed in overcoming the inertia of the system, and which, therefore, are not effective in producing motion. In a body free to move in any direction, the inertia to be overcome by the impressed force before motion can ensue, is obviously as the mass to be moved; but when the body is compelled to turn about an axis, the amount of inertia will obviously vary with its distance from that axis. This resistance to motion about any axis is called *the moment of inertia of the system* with respect to that axis: and we shall see in the next chapter, that this moment is expressed by the sum of the products of all the particles of the body into the squares of their respective distances from the axis of rotation; that is, m, m', m'', &c. denoting the component particles of the mass M, and r, r', r'', &c. their respective distances from the axis of rotation; then, using the rotation already employed at page 57, we shall prove that, with respect to that axis,

$$\textit{moment of inertia} = \Sigma\,(mr^2).$$

At present we shall apply ourselves merely to the determination of this expression in particular cases, as preparatory to the theory

of rotation, to be delivered in the succeeding chapters. But to render the expression suitable for calculation, we must change it into the form $\int r^2 dM$, to which it is shown to be equivalent by precisely the same reasoning as that employed at art. 41, to which the student may turn.

If the whole mass M of the revolving body could be collected into a single point at some distance k from the axis, then the moment of inertia of the system, that is of this point, would be simply k^2 M, and this will be the same as the moment which actually has place, provided we so determine k that

$$k^2 \, \mathrm{M} = \int r^2 d\,\mathrm{M} \;\therefore\; k = \sqrt{\frac{\int r^2 d\,\mathrm{M}}{\mathrm{M}}},$$

so that if we can determine k, which is called the *radius of gyration*, we shall at the same time know the moment of inertia.

It is frequently of consequence to know what mass M′ ought to be placed at any proposed distance k' from the axis, so that the moment of inertia of the point thus loaded may equal that of the mass M. In order to this, we must evidently have $k'^2\mathrm{M}'=k^2\mathrm{M}$ $\therefore \mathrm{M}'=\frac{k^2}{k'^2}\cdot \mathrm{M}$.

1. To determine the radius of gyration of a slender rod of length l revolving about its extremity.

Putting r for any distance measured on the line from the axis, we have by the formula $k=\sqrt{\frac{\int r^2 dr}{l}}=\sqrt{\frac{r^3}{3l}}$, that is for the whole line, or when $r=l$, $k=l\sqrt{\tfrac{1}{3}}=\cdot 57735\, l$.

If the rod revolve about any point in it of which the distances from the extremities are a and b, then, by taking the above integral between the limits $r=-b$, $r=a$, we have, for the line $a+b$,

$$k=\sqrt{\frac{a^3+b^3}{3\,(a+b)}}.$$

2. To determine the radius of gyration of a circle revolving about an axis through the centre, and perpendicular to its plane.

Putting R for the radius of the circle, its area will be $\pi \mathrm{R}^2$, also at the distance r from the axis the area is πr^2, therefore, in this case, $d\,\mathrm{M} = 2\,\pi\, r dr$, so that $k=\sqrt{\frac{2\,\pi \int r^3 dr}{\pi\, \mathrm{R}^2}}=\sqrt{\tfrac{1}{2}\cdot\frac{r^4}{\mathrm{R}^2}}$, that is, when $r=\mathrm{R}$, $k=\mathrm{R}\sqrt{\tfrac{1}{2}}=\tfrac{1}{2}\,\mathrm{R}\sqrt{2}$, and the result would obviously be the same for a right cylinder revolving round its axis. The moment of inertia is $k^2\mathrm{M}=\tfrac{1}{2}\,\pi\,\mathrm{R}^4$.

3. To determine the radius of gyration of a circular annulus or flat ring, the axis of rotation passing through the common centre of the circles perpendicular to their plane.

Calling the radii of the inner and outer circles R and R′, we have, for the area of the annulus, the expression $\pi R'^2 - \pi R^2$

If, instead of the radius R′, we take any variable radius r, intermediate between R and R′, then the expression for the annulus will be $M = \pi r^2 - \pi R^2 \therefore dM = 2\pi r\,dr$

$$\therefore k^2 = \frac{2\int r^3\,dr}{R'^2 - R^2} = \frac{r^4}{2(R'^2 - R^2)},$$

and this, taken between the proposed limits of r, that is from $r=R$ to $r=R'$, is $k^2 = \dfrac{R'^4 - R^4}{2(R'^2 - R^2)} \therefore k = \sqrt{\dfrac{R'^2 + R^2}{2}}$. (1),

when $R=0$ the annulus becomes a circle, and the expression for k agrees with that in last example; when $R'=R$ the expression is $k=R$ . . . . (2), which applies when the annulus becomes merely a circumference. It is easy to see that the expressions (1) would remain the same for a cylindric shell or wheel whose thickness is $R'-R$ revolving about its axis, and the expression (2) applies when the shell is of insensible thickness.

4. To determine the radius of gyration of the circumference of a circle revolving about its diameter.

The distance of any point (x, y) from the axis of rotation is y, the origin being at the centre, and the proposed diameter being taken for the axis of x. Moreover, since

$$\frac{ds}{dx} = \frac{R}{y}, \text{ we have } dM = ds = \frac{R}{y}dx \therefore y^2\,dM = R y\,dx$$

$$\therefore k^2 = \frac{R\int y\,dx}{s} = \frac{R \times \text{area of circle}}{\text{circumference}} = \frac{\pi R^3}{2\pi R} = \tfrac{1}{2}R^2.$$

The student cannot fail to have remarked the similarity between these problems, and those for finding the centre of gravity, and he must further observe that in determining the distance k of the centre of gyration from an axis, the same regard must be paid to the manner in which dM is taken that was necessary in determining the centre of gravity, as the form of this differential will vary with the position of the axis from which k is measured. If the body is a plane area, and if, moreover, the axis from which k is measured is in that plane, then, taking this axis for that of x, and putting dM under the general form $dM = \int dx\,dy$, as at page 64, the expression for k^2M will be, since what we have before called r is now y,

$$k^2 M = \iint y^2\,dx\,dy = \int \frac{y^3\,dx}{3}.$$

Or this expression for the moment of inertia of a plane area, with respect to an axis in it, may be deduced from considering the area to be generated by the ordinate y moving along the axis of x; for as y generates the whole area, so must the momentum of y generate

the whole momentum; in the one case the generatrix is y, and the quantity generated $\int y dx$; in the other case the generatrix (see ex. 1,)

is $\frac{y^3}{3}$ and therefore the quantity generated must be $\int \frac{y^3 dx}{3}$.

5. Applying this formula to the circle revolving about its diameter or axis of x, we have, from the equation of the circle,

$$y^2 = 2rx - x^2 \therefore 2ydy = (2r - 2x)\,dx$$

$$\therefore dx = \frac{ydy}{r-x} = \frac{y\,dy}{\sqrt{r^2 - y^2}}$$

$$\therefore k^2 \mathrm{M} = \frac{1}{3}\int_{-r}^{r} \frac{y^4\,dy}{\sqrt{r^2 - y^2}} = \frac{1}{8}\pi r^4 \text{ (\textit{Int. Calc.}, p. 42)},$$

this is for the semicircle; but if instead of integrating from $y = -r$ to $y = r$, we had integrated from $y = -0$ to $y = 0$, we should have had for the whole circle $k^2 \mathrm{M} = \frac{1}{4}\pi r^4$; as in the whole circle M is twice as great as in the semicircle, k^2 is the same for both, viz. $k^2 = \frac{1}{4} r^2$.

To determine the moment of inertia of a solid of revolution, with respect to the fixed axis, it will be most convenient to view the solid as generated by the motion of a circle which continues always perpendicular to the fixed axis while the centre describes this axis, as explained at page 144 of the Integral Calculus. In this point of view, we may obviously consider the whole moment of inertia to be generated by the moment of inertia of the generating circle; calling the variable radius of this y, the fixed axis being that of x, the generating moment will, by ex. 2, be $\frac{1}{2}\pi y^4$; hence the whole moment generated must be $k^2 \mathrm{M} = \frac{1}{2}\pi \int y^4 dx$.

6. Suppose the solid were a sphere of radius a, then

$$y^2 = 2ax - x^2 \;\therefore\; y^4 dx = (2ax - x^2)^2 dx$$

$$\therefore k^2 \mathrm{M} = \tfrac{1}{2}\pi \int (2ax - x^2)^2 dx = \pi x^3 (\tfrac{2}{3}a^2 - \tfrac{1}{2}ax + \tfrac{1}{10}x^2),$$

and this integral between the limits $x = 0$, $x = 2a$, gives for the whole sphere $k^2 \mathrm{M} = \frac{8}{15}\pi a^5$, $\therefore k^2 = \frac{2}{5}a^2$. In like manner we should find for a cone revolving about its axis $k^2 = \frac{3}{10} r^2$, r being the radius of the base.

For a paraboloid the expression is $k^2 = \frac{1}{3} r^2$.

(156.) It is useful to know how to find the moment of inertia, with respect to an axis, by means of the known moment with respect to some other axis parallel to it. We shall, therefore, now show how this is to be done.

Let AZ be the axis (fig. 112,) for which the moment of inertia is $\int r^2 d\mathrm{M}$, and let A′Z′ be the axis parallel to it for which the moment of inertia $\int r'^2 d\mathrm{M}$ of the same mass M is to be determined. As-

sume AZ to be the axis of z, and AX, AY to be those of x and y; then for every particle m of the body, the corresponding value of Zm or of its projection Am' is $r^2=x^2+y^2$. In like manner, the distance of the two axes being a, if we call the co-ordinates AA′, A′B of this axis α, β, we shall have $a^2=\alpha^2+\beta^2$. Now the distance of the particle m from A′ Z′, that is of the point (x, y) from the point (α, β), is

$$\begin{aligned} r'^2 &= (x-\alpha)^2+(y-\beta)^2 \\ &= x^2+y^2-2\,\alpha\, x-2\,\beta\, y+\alpha^2+\beta^2 \\ &= r^2-2\,\alpha\, x-2\,\beta\, y+a^2, \text{ therefore} \end{aligned}$$

$$\int r'^2\, d\mathrm{M}=\int r^2\, d\mathrm{M}-2\,\alpha\int x\, d\mathrm{M}-2\,\beta\int y\, d\mathrm{M}+a^2\,\mathrm{M};$$

or, putting X and Y for the co-ordinates of the centre a gravity of the system, we have (art. 41,)

$$\int r'^2\, d\mathrm{M}=\int r^2\, d\mathrm{M}+a^2\,\mathrm{M}-2\,\mathrm{M}\,(\alpha\,\mathrm{X}+\beta\,\mathrm{Y}).$$

When the original axis passes through the centre of gravity, then, since Y=0 and X=0, the formula becomes

$$\int r'^2\, d\mathrm{M}=\int r^2\, d\mathrm{M}+a^2\,\mathrm{M}, \text{ or } \mathrm{M}\,k'^2=\mathrm{M}\,(k^2+a^2);$$

hence to the moment of inertia, estimated from an axis through the centre of gravity, we must add the product of the mass by the square of its distance from the new axis, and the sum will be the new moment of inertia.

The foregoing expression for the moment of inertia about any axis, shows that of all axês taken parallel to each other, that which passes through the centre of gravity of the body will be the one in reference to which the moment of inertia is the least; k is in this case called the *principal radius of gyration.*

1. Take a straight line or slender rod l revolving about an axis at the distance a from the middle; then, by the first example of last article, the moment about the middle is $\frac{1}{12}\, l^3$, therefore the moment about the proposed axis is $k^2\,\mathrm{M}=\frac{1}{12}\, l^3+a^2\, l$.

2. Let a circle revolve about any line in its plane.

The moment round a diameter parallel to the line is, by ex. 5, $\frac{1}{4}\pi\, r^4$, and consequently, calling the distance of this diameter from the proposed line a, we have

$$k^2\,\mathrm{M}=\tfrac{1}{4}\,\pi\, r^4+a^2\,\mathrm{M} \;\therefore\; k^2=\tfrac{1}{4}\, r^2+a^2.$$

The result is necessarily the same when the proposed axis is out of the plane of the circle.

3. Required the moment of inertia of a cone revolving about an axis through the vertex, and perpendicular to the axis of the cone (fig. 113).

Let VA be to AB as 1 to n, then calling the distance VA′ of any variable section x, we have A′ P$=nx$ for the radius of that section, and for its moment of inertia the expression, as found in last example, is

$$\tfrac{1}{4}\pi n^4 x^4 + x^2 \mathrm{M} = \tfrac{1}{4}\pi n^4 x^4 + \pi n^2 x^4 = \pi n^2 (\tfrac{1}{4} n^2 + 1) x^4;$$

hence, for a solid generated by this circle, the moment is

$$k^2 \mathrm{M} = \pi n^2 (\tfrac{1}{4} n^2 + 1) \int x^4 dx = \tfrac{1}{5}\pi n^2 (\tfrac{1}{4} n^2 + 1) x^5,$$

which applies to the cone of altitude x.

CHAPTER IV.

ON THE ROTATION OF A SOLID BODY ABOUT A FIXED AXIS.

(157.) Suppose a body of a known form and mass to be freely moveable about a fixed axis, passing through A (fig. 114), and perpendicular to the plane on which the figure is represented.

Suppose an impulse or shock to be given to the body thus circumstanced, and that it is in the direction BC, perpendicular to the plane AB, drawn through the fixed axis; if the impulsion were oblique to this plane we should only have to take account of that component of it which is perpendicular to the plane, because the other being directed towards the fixed axis, would be counteracted by its resistance, and its effect destroyed.

If, when the impulse was given at B, we conceive the mass to have been perfectly free, and to have been concentrated in the point B, or, which is the same thing, if we conceive the impulse to have been directed towards the centre of gravity of an equal mass M, perfectly free, then, v being the velocity communicated, Mv would be the momentum communicated; this expression, therefore, properly represents the intensity of the impulse or the force *impressed* on the systems (150). The effect of this impulse upon the particles of the body, under consideration, is to cause each to describe a similar arc of a circle in the same time, the radius of any one being Am=r; each particle will, therefore, move with the same *angular velocity* about the fixed axis, and this, therefore, may be called the angular velocity of the whole mass; let it be represented by ω. Now the only force impressed on the system is Mv, acting at B, and the forces with which the component particles m, m', m'', &c. move, arise from their mutual connexion with each other; we may, therefore, apply to this inquiry the principle of D'Alembert, and thus obtain an equation between the *impressed* and the *actual* forces. As the particles all have the common angular velocity ω, their actual velocities are $r\omega$, $r'\omega$, $r''\omega$, &c. and, consequently, their moving forces, or momenta, $mr\omega$, $m'r'\omega$, $m''r''\omega$, &c., and these are the forces which acting at the distances r, r', r'', &c. from the axis of rotation, must equilibrate with the force Mv, acting at the distance R=AB, when this latter acts in the opposite direction;

consequently, multiplying each force by its distance from the axis, we must have the equation

$$mr^2\omega+m'r'^2\omega+m''r''^2\omega+\&c.=\text{MR}\,v \;\therefore\; \omega=\frac{\text{MR}v}{\Sigma\,(r^2 m)}.\;(1);$$

which implies that *the angular velocity is equal to the moment of the applied force, divided by the moment of inertia,* just as the linear velocity v, which the same force is competent to produce on an equal mass, is expressed by that force, Mv, divided by the mass moved, by which mass the inertia of the body is always represented: so that as M denotes the resistance to progressive motion, $\Sigma\,(r^2 m)$ denotes the resistance to angular motion. This expression, therefore, is fitly called the moment of inertia of the revolving body.

(158.) Let us now suppose that instead of an impulse giving rotation to the body, every particle of it is actuated by an accelerative force; these forces may be all different, but, as before, we shall consider them as acting in planes perpendicular to the fixed axis. Calling the several forces acting at m, m', m'', &c. F, F$'$, F$''$, &c. and p, p', p'', the perpendiculars from the axis on their directions; the applied motive forces will be mF, m'F$'$, m''F$''$, &c. and ω being the angular velocity, or $r\omega$ the absolute velocity of m, $r\frac{d\omega}{dt}$ will be its acceleration, so that the actual motive forces are $mr\frac{d\omega}{dt}$, $m'r'\frac{d\omega}{dt}$, $m''r''\frac{d\omega}{dt}$, &c.; hence, by D'Alembert's principle, these forces, taken in the reverse direction, balance the former, so that their moments give the equation

$$(mr^2+m'r'^2+m''r'^2)\frac{d\omega}{dt}=m\text{F}p+m'\text{F}'p'+m''\text{F}''p''+\&c.$$

$$\therefore \frac{d\omega}{dt}=\frac{\Sigma\,(\text{F}pm)}{\Sigma\,(r^2 m)}.\;\ldots\,(2);$$

that is, as before, the angular acceleration is equal to the moment of the applied moving forces, divided by the moment of inertia, just as the linear acceleration which the same forces $\Sigma\text{F}\cdot\Sigma m$ is competent to produce in an equal mass, Σm is expressed by that force divided by the mass moved, that is, by Σm.

Let us now apply the result just obtained to the theory of the compound pendulum.

Centre of Oscillation of a vibrating Body.

(159.) When a heavy body vibrates about a horizontal axis, by the force of gravity, the mass is considered as a compound pendu-

lum, and it is an important problem to determine what must be the length of a simple pendulum which shall perform its oscillations in the same time about the same axis of suspension; or rather to determine what simple pendulum must be substituted for the compound mass, in order that the vibrations of both may be the same as regards velocity, acceleration, and time. The foregoing general formula enables us to solve this problem; for as the accelerative force acting upon every particle of the mass is the same, viz. $F=g$, the formula, as applied to this case, may be written

$$\frac{d\omega}{dt}=\frac{g\Sigma\,(mp)}{\Sigma\,(r^2\,m)}\;\ldots\ldots(1).$$

Now let G (fig. 115,) be the centre of gravity of the oscillating body, M, and AB a vertical plane through the horizontal axis of suspension, which axis we here suppose to be perpendicular to the plane of the paper; let GP be perpendicular to this plane when the body is in any proposed position, that is, when the plane AM, through the centre of gravity and perpendicular to the plane of the paper, makes any angle $GAB=\theta$ with the plane AB; then we know, by the theory of the centre of gravity (41), that

$$M\times GP=mp+m'p'+m''p''+,\ \&c.$$

that is, putting a for the distance AG of the axis from the centre of gravity, since $GP=a\sin.\theta$; $M\,.\,a\sin.\theta=\Sigma\,(mp)$, substituting, therefore, this value in the numerator of the expression (1) above, we have

$$\frac{d\omega}{dt}=\frac{M\,.\,a\sin.\theta}{\Sigma\,(r^2m)}=\frac{M\,.\,a\sin.\theta}{M\,(k^2+a^2)}=\frac{a\sin.\theta}{k^2+a^2}\;\ldots\ldots(2),$$

where k represents the radius of gyration, in reference to an axis parallel to that of suspension, and passing through the centre of gravity of the body, and where a is the distance of these two axes. Suppose now that the mass were removed, and that in its stead there were suspended a single particle at the distance l from the axis, then $\Sigma\,(r^2m)$ would become simply l^2m, and in order that the angular acceleration of this simple pendulum may be the same as that of the mass, for which it has been substituted, we must determine l from the equation,

$$\frac{M\,.\,a\sin.\theta}{\Sigma\,(r^2m)}=\frac{ml\sin.\theta}{l^2m}\;\therefore\;\frac{a}{\Sigma\,(r^2m)}=\frac{1}{l}$$

$$\therefore\; l=\frac{\int r^2d\,M}{a\,M}=\frac{a^2+k^2}{a}\;\ldots\ldots(3);$$

this, therefore, expresses the length of the simple pendulum, whose vibrations will agree with that of the compound pendulum: it is equal to the square of the radius of gyration measured from the axis, divided by the distance between the axis and centre of gravity.

That point in the body, which is at the distance l from the axis, is called *the centre of oscillation* of the compound pendulum. There are, it is evident, an infinite number of such centres, or of points at the distance l from the axis, but we here more especially mean that point which is in the line passing through the point of suspension and the centre of gravity of the mass.

As OG is the distance of the centre of gravity G, from the centre of oscillation O, it follows, from the equation (3), that the distance between these two centres is $OG=\frac{k^2}{a}$ . . . . (4); therefore, since so long as the plane of the body's vibration remains the same k must remain the same, it follows that with this condition *the distances between the centres of gravity and oscillation are inversely as the distances between the centre of gravity and point of suspension.*

Moreover, if the centre of oscillation O were made the point of suspension, then to find the corresponding value of l we must put the value (4) for a in (3) which substitution, as it alters not the value of l, shows us that, provided we do not alter the plane of the body's vibration, the *centre of oscillation and point of suspension are convertible;* that is, if we convert the centre of oscillation into the point of suspension, the point of suspension will become the centre of oscillation.

Also the distances of the axis of suspension from the centres of gravity, of gyration, and of oscillation, are in continued proportion, for $a : \sqrt{\{a^2+k^2\}} :: \sqrt{\{a^2+k^2\}} : a+\frac{k^2}{a}$. We may, likewise, infer from the value of l, what the distance a of the centre of gravity from the axis must be, in order that for the same plane of vibration the time in which the body performs its oscillations may be the least possible; for as the equivalent simple pendulum will vibrate the quicker the shorter it is, we shall merely have to determine a so that we may have

$$l=a+\frac{k^2}{a}=a \textit{ minimum}. \quad \therefore \frac{dl}{da}=1-\frac{k^2}{a^2}=0 \therefore a=k;$$

so that the axis of suspension must pass through the principal centre of gyration. Several other particulars might be deduced from the foregoing investigation, but we shall mention here but one more, which is that if the compound mass consist of several distinct bodies, the centre of oscillation of the whole will be found by taking the continued product of each mass into the respective distances of its centres of oscillation and of gravity, from the axis of suspension, adding the products together, and dividing the sum by the product of the whole system into the distance of the common centre of gravity from the axis.

For calling the several masses M, M′, &c. and the length of the equivalent pendulum L, we have, by the preceding theory,

$$\left.\begin{array}{l} l=\dfrac{\mathrm{M}\,(a^2+k^2)}{\mathrm{M}\,.\,a} \\ l'=\dfrac{\mathrm{M}'\,(a'^2+k'^2)}{\mathrm{M}'\,.\,a'} \\ \&c. \quad \&c. \end{array}\right\} \therefore \Sigma\,(l\,.\,\mathrm{M}\,.\,a)=\Sigma\,\{\mathrm{M}\,(a^2+k^2)\}.$$

As the second member of this equation expresses the moment of inertia of the whole system, it follows that

$$\mathrm{L}=\frac{\Sigma\,(l\,.\,\mathrm{M}\,.\,a)}{\Sigma\,(\mathrm{M}\,.\,a)} \quad \ldots\ldots (5);$$

which establishes the proposition, since the denominator of this fraction is equal to the whole compound mass into the distance of its centre of gravity from the axis. This same formula will, obviously, serve for any vibrating mass when we find it convenient to consider it in separate parts.

(160.) We shall now give a few examples of the determination of the centre of oscillation in different bodies.

1. To determine the centre of oscillation of a slender rod or straight line suspended at any point.

Let a, b, be the lengths, on contrary sides, of the point of suspension, then (155) for k^2 we have, $k^2=\dfrac{\frac{1}{3}\,(a^3+b^3)}{a+b}$.

Again, the centre of gravity being in the middle of the line, its distance from the point of suspension is $\frac{1}{2}\,(a-b)$; hence (3),

$$l=\frac{\frac{1}{3}\,(a^3+b^3)}{\frac{1}{2}\,(a^2-b^2)}=\frac{2\,(a^2-ab+b^2)}{3\,(a-b)}.$$

If the rod is suspended at its extremity, then $b=0$, and $l=\frac{2}{3}a$, or two thirds the length. If it is suspended at its middle, then $a=b$, and $l=\infty$, that is, the centre of oscillation is at an infinite distance, and, therefore, to perform one vibration would require an infinite length of time, and this is the same as saying that no vibration at all could take place; indeed, in whatever position about the centre of motion the rod be placed, it would obviously rest there, seeing that its centre of gravity would be supported.

If $b=\frac{1}{2}a$, then $l=a$, or two thirds the whole length, the same distance as when the rod is suspended at its extremity; so that in both these cases the oscillations will be performed in the same time, as, indeed, ought to be the case, because the centres of suspension and of oscillation are merely interchanged, (p. 194.)

2. To determine the centre of oscillation of an angular pendulum (fig. 116,) composed of two equal slender rods AB, AC.

Bisect the arms AB, AC, in g and γ; these points will be their

centres of gravity; bisect the line joining them in G, and this will be the centre of gravity of the system, and AG=Ag sin. g, or AG $=\frac{1}{2}a\cos.\theta$.

Again, the distance of A from the centre of oscillation of the part AB or of AC, is equal to $\frac{2}{3}$ AB$=\frac{2}{3}a$, therefore, by the formula (5),

$$L=\frac{2\left(\frac{2}{3}a\,.\,a\,.\,\frac{1}{2}a\right)}{2a\,.\,\frac{1}{2}a\cos.\theta}=\frac{2}{3}a\sec.\theta;$$

hence because L, the length of the equivalent simple pendulum, increases with θ, it follows that the time of vibration of an angular pendulum may be increased without limit, by merely increasing the angle between the arms; when $\theta=0$, that is, when the arms close, and form but one rod $=a$, the corresponding value of L is $\frac{2}{3}a$, and when $\theta=90°$ or when the arms open into a straight line, L is infinite.

When the time of vibration is given, we may easily determine the corresponding angle of the arms when their lengths are known, for, from the given time, L will be determined, and thence

$$\sec.\theta=\frac{3L}{2a}.$$

Thus if the time is one second, then L $=39\frac{1}{8}$, and if $a=15$ inches sec. $\theta=75°\cdot11'\frac{1}{2}$ $\therefore\ \angle$ BAC$=150°\cdot23'$.

3. To determine the centre of oscillation of a sphere.

Let r be the radius of the sphere, then the mass is $\frac{4}{3}\pi r^3$, and the square of the principal radius of gyration is (p. 189-190,) $k^2=\frac{2}{5}r^2$; hence

$$l=a+\frac{k^2}{a}=a+\frac{2}{5}\cdot\frac{r^2}{a}\ \ldots\ (1);$$

a being the distance of the centre of the sphere from the point of suspension.

If the axis of suspension were a tangent to the sphere, we should have $r=a$, therefore, in that case, $l=r+\frac{2}{5}r$.

From the expression (1) we get for a, when l is given,

$$a=l\tfrac{1}{2}\pm\sqrt{\left\{\tfrac{1}{4}l^2-\tfrac{2}{5}r^2\right\}};$$

so that a sphere may be suspended at two different distances from the centre, and yet vibrate in the same proposed time; if, however, $\frac{1}{4}l^2=\frac{2}{5}r^2$, that is, if the time of vibration is to be that which belongs to the simple pendulum, $l=2r\sqrt{\frac{2}{5}}$, then there is but one suitable distance for the centre from the axis of suspension, viz. the distance $a=r\sqrt{\frac{2}{5}}$.

4. Suppose the bob of a clock pendulum to consist of two spheric segments joined at their bases: to determine the distance of the centre of oscillation from the centre of the bob.

Let r be the radius of the sphere, and x the height of one of the segments, then the moment of inertia of the two segments is (ex. 6,

p. 190), $Mk^2 = 2\pi x^2 (\frac{2}{3} r^2 - \frac{1}{2} rx + \frac{1}{10} x^2)$, and the volume of the two segments is

$$M = 2\pi x^2 (r - \tfrac{1}{3} x) \therefore \frac{k^2}{a} = \frac{x(\frac{2}{3} r^2 - \frac{1}{2} rx + \frac{1}{10} x^2)}{a(r - \frac{1}{3} x)}$$

and this is the distance of the centre of gravity of the bob from the centre of oscillation.

If instead of the radius r of the sphere, the radius r' of the base of each segment is given; then since $r'^2 = (2r - x)x$

$$\therefore r = \frac{x^2 + r'^2}{2x} \text{ and, by substitution, } \frac{k^2}{a} = \frac{r'^4 + \frac{1}{3} r'^2 x^2 + \frac{1}{15} x^4}{a(3r'^2 + x^2)},$$

a added to either of these expressions will give the distance of the centre of oscillation from the point of suspension.

5. To determine the centre of oscillation of a cone suspended at its vertex.

Putting (as in ex. 3, page 191), x for the altitude of the cone, and nx for the radius of its base, we have, for the moment of inertia, the expression $\frac{1}{5} \pi n^2 x^5 (\frac{1}{4} n^2 + 1)$, also the distance of the centre of gravity from the vertex is (46,) $\frac{3}{4} x$, consequently,

$$l = \frac{\frac{1}{5} \pi n^2 x^5 (\frac{1}{4} n^2 + 1)}{M . \frac{3}{4} x} = \frac{\frac{1}{5} \pi n^2 x^5 (\frac{1}{4} n^2 + 1)}{\frac{1}{3} x . \pi n^2 x^2 . \frac{3}{4} x} = \tfrac{1}{5} n^2 x + \tfrac{4}{5} x.$$

6. In a similar manner is found for the centre of oscillation of a circle vibrating edgewise, and suspended at the distance a from its centre, $l = a + \frac{r^2}{2a}$.

7. And when the circle vibrates flatwise $l = a + \frac{r^2}{4a}$.

On the Centre of Percussion.

(161.) When a solid body revolves about a fixed axis, the force with which it would strike a fixed obstacle would vary with the situation of the point of impact, as also would the shock received by the immoveable axis; but there is a point at which, if the impact take place, the obstacle will receive the whole force of the moving body, and the axis will receive none, so that, if at the instant of impact the axis were to be annihilated, the body would still remain at rest. This point is called the *centre of percussion*. In order to ascertain the situation of this centre, let us refer to the expression (1) for ω, ω being here considered to represent the angular velocity at the instant of impact; let also the moving force due to the impact, and which we before represented by Mv, be called f, and the distance of the direction of impact from the axis D, the formula referred to then gives $f = \frac{\omega \Sigma (r^2 m)}{D}$; this ex-

presses, therefore, the force of percussion at the distance D. Now if this force is equal to that of the whole moving mass, no portion of it will be expended in straining the axis, and D will in that case measure the distance of the centre of percussion from the axis.

To determine what the whole force of the revolving body is, we must add together the forces due to the component particles; that due to any one, m, is $mr\omega$; hence, calling the whole force F, we have

$$F=\omega\,\Sigma\,(rm) \therefore \omega\,\Sigma\,(rm)=\frac{\omega\,\Sigma\,(r^2m)}{D} \therefore D=\frac{\Sigma\,(r^2m)}{\Sigma\,rm)},$$

consequently the distance of the centre of percussion from the axis is equal to the distance of the centre of oscillation from the axis; if, therefore, the impact be perpendicularly directed to the plane passing through the axis of suspension and centre of gravity, then the centres of percussion and oscillation will be in the straight line parallel to the axis of suspension.

If the plane through the centre of gravity, and perpendicular to the fixed axis, divide the body symmetrically, it is plain that whatever force directed in this plane strike the body, the axis may suffer a direct shock, but it will not be twisted; in such bodies, therefore, the centre of percussion coincides with the centre of oscillation, because at this point the impact will neither strain nor twist the axis. But in other cases, impact at the centre of oscillation, although it would occasion no direct strain on the axis, may yet tend to twist it, as it is easy to conceive, (see *Dr. Gregory's Mechanics*, vol. 1. p. 300; and *Francœur's Mècanique*, p. 352.)

Between the centre of gyration of a body revolving about a fixed axis and the centre of percussion, we may remark this difference, viz. the centre of gyration is the point in which, if the whole revolving mass were concentrated, the same angular motion would be generated by any force as before, and therefore, an obstacle meeting the line which connects this point with the axis of motion, will be struck with the same force as it would be by the revolving mass, at whatever point of the line the impact take place. But the centre of percussion is that particular point which would strike an obstacle with the whole force of the revolving mass.

Of the Centre of Spontaneous Rotation.

(162.) Intimately connected with the centre of percussion is that of *spontaneous rotation*.

We have just seen that when a body revolves on an axis, there exists a point at which a fixed obstacle would receive the whole of its force, so that if this same force were to strike the body when at

rest in the same point, it would produce in it a rotatory motion round the same axis, even though that axis had been removed; the axis about which a quiescent body, when struck in a direction not passing through the centre of gravity, thus spontaneously revolves, is called the axis of spontaneous rotation.

Instead then of considering the body which receives the impulsion to be retained by a fixed axis, let us suppose it to be perfectly free; or, to render the inquiry the more general, let us first consider a system of material particles m, m', m'', &c. entirely free and unconnected, and moving with the parallel velocities v, v', v'', &c. and let us inquire what will be the motion of the centre of gravity of the system.

Through this centre conceive a plane parallel to the directions of the impulsions; as, at the commencement of the motion, the sum of the moments of m, m', m'', &c. with respect to this plane, is 0, (p. 59,) and as the bodies preserve their respective distances from the plane throughout the motion, it follows that the sum of the moments, with respect to this plane, must be the same at every instant; hence, the sum being always 0, the centre of gravity of the system must necessarily move always in this plane. Conceive another plane also parallel to the directions of the impulsions, and passing, in like manner, through the centre of gravity of the system, but making an angle with the former plane, then, as before shown, the centre of gravity will always move in this plane, it must, therefore, describe the line of intersection of these planes, that is, *the centre of gravity of the system describes a straight line parallel to the directions of the impressed velocities.*

Conceive now a plane perpendicular to the directions of the velocities, and design by e, e', e'' &c. the distances of the points m, m' m'', &c. from this plane at the commencement of motion: at the end of the time t'', their distances will be $e+vt$, $e'+v't$, $e''+v''t$, &c.; let us take the moments with respect to this plane, then a, x being the respective distances of the centre of gravity from the plane at the commencement of motion, and at the end of the time t'', we shall have the following equations (41)

$$(m+m'+m''+\&c.)\ a=me+m'e'+m''e''+\&c.$$
$$(m+m'+m''+\&c.)\ x=m(e+vt)+m'(e'+v't)+\&c.$$

Subtracting the first from the second we have

$$(m+m'+m''+\&c.)\ (x-a)=(mv+m'v'+m''v''+\&c.)\ t,$$

which shows that the space $x-a$, described by the centre of gravity, is proportional to the time; hence *the centre of gravity of the system moves uniformly.* It must be remembered, that in the foregoing equations those values of e, e', &c., of v, v', &c., are taken negatively which are measured in opposite directions to those considered as positive.

As $\frac{x-a}{t}$ is the velocity of the centre of gravity, it follows, that if the whole mass of the system were concentrated there, its momentum or quantity of motion would be

$$(m+m'+m''+\&c.)\frac{x-a}{t}=mv+m'v'+m''v''+\&c.$$

by the equation just deduced; hence the second member of this equation represents the intensity of the impulsion that must be applied to the whole mass of the system, when concentrated in the centre of gravity, to make that centre move, as it actually does; we conclude, therefore, that *the centre of gravity moves with the same velocity as if all the impulsions were immediately impressed on it*, or, which is the same thing, *the centre of gravity moves as if the whole mass of the system were concentrated in it, and all the forces were applied to it in directions parallel to those they really take.*

If the impressed velocities were not parallel, the same thing would also have place. For if we decompose each of them into three others parallel to rectangular axes, we may apply to each of the three groups of parallel velocities the foregoing reasoning, and thence infer that the centre of gravity would move in the direction of each axis, as if the forces parallel to that axis were immediately applied to that centre; and therefore its motion in space being compounded of these, it moves as if all the impulsions distributed through the system were directly applied to the centre of gravity or to the whole mass concentrated there.

Let us now suppose the bodies in the system to be invariably connected together, as in the case of one solid mass.

Let P, P′, &c. represent the impulsive forces which act on the several bodies, decompose each force into two others; the one due to the motion which it actually produces, and the other due to the motion destroyed on account of the mutual connexion of the bodies in the system; so that F, F′, &c. may be the forces which produce their full effect, and f, f', &c. those which are destroyed by the mutual action of the parts of the system; thus F and f will be the components of P; F′, and f' the components of P′; &c. In virtue of the forces F, F′, &c., which are fully effective, the motion must be the same as if these forces only acted on the system, all connexion between its parts being destroyed, so that from what has been proved above, the centre of gravity ought to move as if all the forces F, F′, &c. were immediately applied to it, in directions parallel to those which they actually take. As to the forces f, f', &c. they are mutually destroyed when they act upon the several parts of the system, and consequently satisfy the six equations (6), (7), page 79;

but when transported, parallel to themselves, to the centre of gravity, they ought, for much greater reason, to be mutually destroyed, since then the equations (4) page 28, suffice to establish their equilibrium. Hence *the centre of gravity moves as if the several impulsions were immediately applied to it.*

(163.) Let us now examine the motion of a body which receives an impulsion that does not pass through the centre of gravity; the motion of translation would, we know, from what is proved above, be the same as if the impulsion were applied in a parallel direction to the centre; but, beside this, there would be impressed a motion of rotation, precisely the same as would have been impressed by the same force if a fixed axis had passed through the centre of gravity. This double motion, arising from a single impulsion, may be at once shown to take place as follows. Let P (fig. 117) represent the impulsive force, and, perpendicular to its direction, draw GA from the centre of gravity; at an equal distance GB, on the other side of G, let two forces $\frac{1}{2}$P and $\frac{1}{2}$P, equal and opposite to each other be applied, these will have no effect on the system, so that we may consider the motion which the body actually takes, in consequence of the single force P, to be the result of the three forces acting as in the figure at A and B. The motion of translation is due to the force P, considered as acting at G, or, which is the same thing, this motion is due to the force $\frac{1}{2}$ P acting at A, and to $\frac{1}{2}$ P acting at B, in direction BS; the remaining forces, therefore, that is the force $\frac{1}{2}$P at A, and the opposite force $\frac{1}{2}$ P at B, in direction BR, are those to which the rotatory motion is due; the tendency of these forces is to turn the body about G, being symmetrically situated with respect to it, and the value of the forces to produce this effect is at A, $GA\times\frac{1}{2}P$, and at B, $GB\times\frac{1}{2}P$, and as these forces turn the system in the same direction their whole effect is

$$GA\times\tfrac{1}{2}P+GB\times\tfrac{1}{2}P=GA\times P;$$

which is the effect due to the impressed force P to turn the body about a fixed axis through s.

It follows, therefore, that *when a body is acted upon by any impulsive forces, of which the resultant does not pass through the centre of gravity, the body will have, in consequence, a double motion;* 1, *the centre will move as if the forces were immediately applied to it;* 2, *the body will turn as if this centre were absolutely fixed.*

Let P (fig. 118) be the momentum, or quantity of motion, impressed on the body, r its distance OG from the centre of gravity of the body M; then, for the velocity of translation due to this force, we have $v=\frac{P}{M}$. Again, for the angular velocity ω due to the same

force P, acting at the distance GO from the centre of motion, we have (157) $\omega=\frac{\mathrm{P}r}{\mathrm{M}k^2}$; consequently, the absolute velocity of any point in the body is compounded of these two, viz.

$$\left.\begin{array}{r} \textit{progressive velocity, } v=\frac{\mathrm{P}}{\mathrm{M}} \\ \textit{angular velocity, } \omega=\frac{\mathrm{P}}{\mathrm{M}}\cdot\frac{r}{k^2}=\frac{vr}{k^2} \end{array}\right\}\ldots\ldots(1);$$

r being the perpendicular distance of the centre of gravity of the mass M, from the direction of the applied force P, and k being the principal radius of gyration. These results may be expressed in words, as follows, viz. *the progressive velocity is equal to the moving force, divided by the mass of the body, and the angular velocity is equal to the moment of the force divided by the moment of inertia.*

If, however, the body is not free to revolve about its centre of gravity, but is constrained to turn about some other point moving uniformly, the angular velocity will be different. It will be easy, however, to estimate it, for as the tendency to turn about the centre of gravity is the same on whichever side of it the impulsion be given, provided only it act at the same distance from it and in contrary directions, we may obviously consider the angular motion which accompanies the progressive motion of the point, to be the result of a force acting in a direction opposed to the progressive motion, and at the opposite side of the centre of gravity, but at the same distance from it as the centre of motion. The value of the impulse to which the motion of the body is due, will be known from knowing the progressive velocity of the centre of motion and the mass moved. We shall give an illustrative example of this hereafter (at prob. IV., chap. VII.); at present we consider the body as entirely free, in which case we observe the following particulars.

At the instant the impact is given, the point O departs in the direction Oh, its initial velocity being equal to the sum of its progressive and angular velocities as their directions coincide, and the same is, obviously, the case with any other point in GO. With the points in GO′, on the other side of G, it is different, for although they have the same angular velocities as the corresponding points in GO, yet, turning in the contrary direction, their absolute velocity of any one of them is equal to the difference between its progressive and angular velocities; that is to say, every point O′ in the line GO′ has, in virtue of the angular motion of the system, a velocity backward, and, in virtue of the progressive motion of the system, a velocity forward; this latter is the same for all the points in the body, and equal to that of the centre of gravity, but the

backward velocity of every point in GO′ varies with its distance from G, being 0 at G, and increasing regularly as the distance increases; there must in consequence be some point in GO′, either within or without the body of which the velocity backward is precisely equal to the velocity forwards; this point then is for an instant at rest while all the other points of the body are in motion; so that the whole system, when the initial motion is given, turns spontaneously round it; this point is hence called the *centre of spontaneous rotation.* Its situation is readily determined from the condition which characterizes it, which is, that calling its distance from G, r', and progressive velocity of the system v, $v-r'\omega=0 \therefore r'=\frac{v}{\omega}$, or, from equation (1) above, $r=\frac{k^2}{r}$; hence, if C (fig. 119) be the point thus determined, its property, with respect to the point of impact O, is that $OC=r+\frac{k^2}{r}$ .... (2); which proves (159) that *the centre of spontaneous rotation coincides with the centre of suspension corresponding to the point of percussion, considered as the centre of oscillation,* and is entirely independent of the intensity of the applied force.

(164.) In order to determine at what distance GO, from the centre of gravity, the impulsion must have been given to produce the actual progressive and rotatory motions observed in any body, we have, from the equation (1) above, $r=\frac{k^2\omega}{v}$; or, if V be the rotatory velocity of any point at the distance R from the centre of gravity, then since $\omega=\frac{V}{R}$, we have $r=\frac{k^2}{R}\cdot\frac{V}{v}$. (1).

Applying this to the double motions of the planets, we may determine at what distance, from the centre of each, the original impulsion must have been impressed by the hand of the Creator to cause their actual motions of progression in space and rotation on their axes.

Taking the earth for example, we know that it performs its revolution on its axis in a sidereal day, by which rotatory motion every point on the equator passes over about 25020 miles.

Also its orbit, or a path of about 596904000 miles, is passed over by its progressive motion in 366 sidereal days, hence the ratio

$$\frac{V}{v} \text{ is here } 1 \div \frac{V}{v} = \frac{596904000}{25020 \times 366} = 65.3\,;$$

and, considering the earth a sphere, we have (p. 189) $k^2=\frac{2}{5}R^2$; hence, by substituting these values in the formula (1), we have for the distance r from the centre of the sphere at which the impulse

was given $r = \frac{R}{163 \cdot 2}$; that is, about the $\frac{1}{163}$ part of the radius distant from the centre.

It is very probable that not only the planets but that also the sun may thus derive its motion from a single primitive impulse, and if so, he, in common with the planets, must also have a progressive motion in space; this cannot, indeed, be rigorously proved. "But," to use the words of Dr. Robison, as quoted by Professor Gregory, "the very circumstance of his having a rotation in 27d. 7h. 47m. makes it very probable that he, with all his attending planets, is also moving forward in the celestial spaces, perhaps round some centre of still more general and extensive gravitation; for the perfect opposition and equality of two forces necessary for giving a rotation without a progressive motion, has odds against it of infinity to unity.* This corroborates the conjectures of philosophers, and the observations of Herschel and other astronomers who think that the solar system is approaching to that quarter of the heavens in which the constellation Aquila is situated."

CHAPTER IV.

PROBLEMS ILLUSTRATIVE OF THE PRECEDING THEORY.

(165.) We shall now proceed to illustrate the theory delivered in the two preceding chapters, by showing its practical application in a few miscellaneous problems. Several of these are those selected by Mr. Barlow in his Treatise on Mechanics, in the Encyclopædia Metropolitana.

PROBLEM I.—Let AB (fig. 120) denote an axle turning on two fulcrums at A and B; RS a wheel of given diameter to which is attached, by a cord wound round its circumference, a given weight W; conceive p, p', p'', p''', to be given weights, fixed to the axes by inflexible lines or wires, Cp, Cp', &c. and let it be required to determine the circumstances of the motion of the descending weight.

We shall, in the first place, consider this problem under its most simple form, viz. we shall suppose the wheel RS, the axle AB, and

* It does not appear to us, however, that any weight should be attached to this assertion, founded on the doctrine of chances, and which can strictly apply only to the case of two impulsive forces, directed at random towards opposite parts of a spherical body. Whatever primitive motions the Almighty may have designed to impress on the sun, the impulses it must have received could not fail to be those precisely competent to produce the intended effect.

the lines or wires Cp, Cp', &c. as divested of inertia, or as offering no resistances to angular motion; so that W and the four masses p, p', p'', p''', will be the only weights in the system, the latter being all equal, placed at equal distances from the axle C, and at right angles to each other.

Let Cp, Cp' &c.$=r$; the radius of the wheel $=r'$; the sum $p+p'+p''+p'''=\text{P}$, and the given weight $=\text{W}$; then the moment of inertia of P will be $r^2\frac{\text{P}}{g}$, and the moment of W, the moving force, will be $r'\text{W}$, consequently the acceleration of W, being equal to r' times the angular acceleration, will be $r'^2\text{W} \div r'^2\frac{\text{W}}{g} + r^2\frac{\text{P}}{g}$ that is, $\text{F}=\frac{r'^2\text{W}}{r'^2\text{W}+r^2\text{P}}g$,; and the velocity of W, after any time t'', will be

$$v=\text{F}t=\frac{r'^2\text{W}}{r'^2\text{W}+r^2\text{P}}gt;$$

and the corresponding space descended will be

$$s=\tfrac{1}{2}vt=\frac{r'^2\text{W}}{r'^2\text{W}+r^2\text{P}}\,\frac{gt^2}{2};$$

and thus the circumstances of W's motion are all known.

Let us now take into consideration the inertia of the wheel and axle; call the weight of the former W′ and that of the latter W″, also let r'' be the radius of the axle; then (page 187,)

$$\text{the moment of inertia of the wheel} = \tfrac{1}{2}r'^2\frac{\text{W}'}{g}, \text{axle} = \tfrac{1}{2}r''^2\frac{\text{W}''}{g},$$

the square of the radius of gyration being in both cases $k^2=\frac{1}{2}r^2$.

The motive force being still the same as before, we have, for the acceleration of W,

$$\text{F}=\frac{r'^2\,\text{W}}{r'^2\,\text{W}+\frac{1}{2}r'^2\,\text{W}'+\frac{1}{2}r''^2\,\text{W}''+r^2\,\text{P}}\,g.$$

The accelerating force being thus known, the space, velocity, time, &c. are determined by the usual formulas for constant forces.

If we suppose the system of small weights p, p', p'', &c. to be replaced by a solid body of revolution, as in fig. (121) the principles of the calculation will be still the same; for the moment of inertia of the solid P will as before be $k^2\frac{\text{P}}{g}$, k being the principal radius of gyration as measured from the geometrical axis. Thus in fig. 121, let P denote a sphere whose radius is 3 feet, and weight 500 lbs.; the weight W=50 lbs., and the radius of the wheel 6 inches, or $\frac{1}{2}$ a foot, and of which the weight, as well as that of the axle, are supposed inconsiderable with respect to the other parts of the system;

and let it be required to determine the time in which the weight W will pass through any given space, as, for example, 50 feet.

In the sphere $k^2=\frac{2}{5}r^2$; hence the expression for the acceleration of W is

$$F=\frac{r'^2W}{r'^2W+\frac{2}{5}r^2P}\,g=\frac{\frac{1}{4}\times50}{\frac{1}{4}\times50+\frac{2}{5}\times9\times500}\,32\tfrac{1}{6}=\frac{402\frac{1}{12}}{1812\frac{1}{2}}\text{ feet}$$

therefore, $s=\frac{402\frac{1}{12}}{1812\frac{1}{2}}t^2=50 \therefore t=\sqrt{\frac{50\times870}{193}}=15$;

hence the time of descent is 15″.

Problem II.—Let ABC (fig. 121) represent a wheel and axle, its weight w, having a given weight W applied to the circumference of the axle, and P applied to the circumference of the wheel in order to raise W; it is required to assign the space described by the elevated weight W from rest, in any given time.

Let the radius of the axle be r, that of the wheel R, and the principal radius of gyration of the wheel k; then, for the moment of inertia of the whole system, we shall have the expression

$$k^2\frac{w}{g}+R^2\frac{P}{g}+r^2\frac{W}{g}.$$

Now the actual weight, which, applied at the point D, gives motion to this, is not the whole weight P, since part of this is employed in *balancing* the weight W; to know what this part is, we have, by calling it P′, the equation $Wr=P'R \therefore P'=\frac{Wr}{R}$ so that only

the weight $$P-\frac{Wr}{R}=\frac{PR-Wr}{R}$$

is actually employed in *moving* the system, and as this weight acts at the point D we must multiply it by R to obtain its moment, and dividing R times this by the mass moved, or by the whole inertia we have, for the acceleration of P, the expression

$$F=\frac{PR^2-WRr}{k^2w+R^2P+r^2W}\,g \ldots . (1).$$

Now as the acceleration of P is to the acceleration of W as the radius R to the radius r, we have

$$R:r::\frac{PR^2-WRr}{k^2w+R^2P+r^2W}\,g:\frac{PRr-Wr^2}{k^2w+R^2P+r^2W}\,g \ldots . (2),$$

which expresses the acceleration of the ascending weight.

If $R=r$ the acceleration of either weight will be

$$F=\frac{P-W}{k^2w+R^2(P+W)}\,R^2g.$$

It should be remarked, that if the mass moved, W, have no weight

but inertia only, or rather if its weight is otherwise supported, and its inertia only has to be overcome by the machine, as for instance when it is to be moved along a perfectly smooth horizontal plane, then, in the numerators of the foregoing expressions, we must put $W=0$.

PROBLEM III.—Let ABC (fig. 121,) represent a wheel and axle of given weight moveable about a horizontal axis which passes through S; and suppose a known weight W is applied to the circumference of the axle, to be raised by a given force P applied to the circumference of the wheel; to assign the proportion of the radii of the wheel and axle, so that the time in which the weight W ascends through any given space shall be a minimum.

Since the ratio only of the radii of the wheel and axle is required, let the radius of the axle be r, and that of the wheel xr; the weight of the wheel w and k as before, the principal radius of gyration of the wheel, and of which the value we know is (p. 188) $k^2=\frac{1}{2}x^2r^2$. Then, substituting xr for R and $\frac{1}{2}x^2r^2$ for k^2, in equation (2) of the preceding problem, we have, for the acceleration of W, the expression $\frac{Px-W}{(\frac{1}{2}w+P)x^2+W}g$, and consequently

$$s=\frac{\frac{1}{2}(Px-W)}{(\frac{1}{2}w+P.)x^2+W}gt^2 \therefore t=\sqrt{\left\{\frac{s(\frac{1}{2}w+P)x^2+Ws}{\frac{1}{2}g(Px-W)}\right\}}.$$

This expression is to be a minimum, and consequently the quantity under the radical will be a minimum; therefore, dividing this by the constant $\frac{s}{\frac{1}{2}g}$, and putting for brevity p for $\frac{1}{2}w+P$, we have

$$\frac{px^2+W}{Px-W}=a\ minimum;$$

hence, by differentiating,

$$2px(Px-W)-P(px^2+W)=0 \therefore Ppx^2-2Wpx=PW$$

$$\therefore x=\frac{W}{P}\pm\sqrt{\left\{\frac{W^2}{P^2}+\frac{W}{p}\right\}}\ \ldots\ (1).$$

If the weight of the wheel be too inconsiderable to deserve notice, then $p=P$, and in this case $x=\frac{W\pm\sqrt{\{W^2+PW\}}}{P}$ (2), and if, moreover, $P=W$, we have $x=1\pm\sqrt{2}$.

Suppose, for example, ABC to represent a cylindrical wheel, the radius of which is required, but of which the weight is 20 lb; and let the radius of the axle be 1 inch; the weight, W, 100 lb., and the weight P, 33 lb.; to find the radius of the wheel.

Here $\frac{1}{2}w+P=p=43$lb., therefore, by equation (1),

$$x=\frac{100}{33}\pm\sqrt{\left\{\frac{100^2}{33^2}+\frac{100}{43}\right\}}=6\,.\,43 \text{ nearly}.$$

Consequently, since the radius of the axle is 1 inch, the radius of the wheel must be 6·43 inches.

For other such problems as this, the student may consult Dr. Gregory's chapter on the "Maximum Effects of Machines."

PROBLEM IV.—In the wheel and axle, when a given weight P acting at the distance R raises a weight W acting at the distance r from the geometrical axis, it is required to assign the pressure sustained by the axis, the weight of the wheel and axle, and the friction of the cord not being considered.

Suppose P and W to be at their respective extremities of the horizontal line passing through the centre of motion, and in that situation let O and G be the respective distances of the centres of oscillation and gravity from the centre of motion; then, since the angular velocity of any revolving system is the same as if its whole mass were concentrated in its centre of oscillation, we may consider P+W to be placed at the distance O from the centre of motion; and, since the force of any point in the revolving system is proportional to its distance from the axis of motion, we have

$$\mathrm{O} : \mathrm{G} :: \mathrm{P} + \mathrm{W} : \frac{\mathrm{G}}{\mathrm{O}}(\mathrm{P}+\mathrm{W}),$$

which is the force or pressure with which the centre of gravity descends; or in fact the force with which the whole mass descends. Part of the whole pressure P+W of the system is thus supported by the axle, and the other part, which we have just seen is $\frac{\mathrm{G}}{\mathrm{O}} \times (\mathrm{P}+\mathrm{W})$, is employed in producing the motion which actually has place; consequently, that part of the pressure sustained by the axle must be

$$\mathrm{P}+\mathrm{W}-\frac{\mathrm{G}}{\mathrm{O}}(\mathrm{P}+\mathrm{W}).$$

It remains, therefore, to find the values of G and O, which are

$$\mathrm{G} = \frac{\mathrm{PR}-\mathrm{W}r}{\mathrm{P}+\mathrm{W}}, \quad \mathrm{O} = \frac{\mathrm{PR}^2+\mathrm{W}r^2}{\mathrm{PR}-\mathrm{W}r};$$

hence, by substitution, we have for the pressure p,

$$p = \mathrm{P}+\mathrm{W}-\frac{(\mathrm{PR}-\mathrm{W}r)^2}{\mathrm{PR}^2+\mathrm{W}r^2} = \mathrm{PW}\frac{(\mathrm{R}+r)^2}{\mathrm{PR}^2+\mathrm{W}r^2}.$$

If $\mathrm{R}=r$, as in the case of the single fixed pulley, then the pressure is $p = \frac{4\mathrm{PW}}{\mathrm{P}+\mathrm{W}}$.

PROBLEM V.—LET A, B (fig. 122,) represent a single moveable pulley by means of which the power P elevates the weight W;

then, having given P and W, together with the weights of the equal cylindric pulleys A and B, it is required to assign the space which the descending weight P describes in a given time, the weight of the moveable pulley being included in the weight W.

Let us refer the whole inertia of the system to the point p, so that we may consider the force which moves P to be burdened with the mass of P, and with the additional mass representing the inertia of the other parts of the system, this mass being all accumulated at p, or, which is the same thing, incorporated in P.

In the first place the inertia of the pulley A, whose weight call Q, is the same as that of half its mass placed at p, (see page 188.)

In like manner the inertia of the pulley B is the same as that of half its mass placed at q; or, since the rotatory velocity of B's circumference is only half the velocity of A's circumference, the mass of half B at q has the same inertia as the same mass placed at half the distance Op, or finally as $\frac{(\frac{1}{2}\,Op)^2}{Op^2}$ times the same mass placed at p, (p. 188.) Hence, as far as the inertia of the pulleys is concerned, the equivalent mass to be placed at p is $\frac{1}{2}\,.\,\frac{Q}{g}+\frac{1}{8}.\frac{Q}{g}$.

Again, as the velocity of W is half that of P, it moves as if it hung at half the distance Op' from the centre of the pulley A, where, as in the case of the pulley B, it would offer the same inertia as $\frac{1}{4}$ its mass placed at p' or p; hence the inertia of the weight is represented by the mass $\frac{P}{g}+\frac{W}{4g}$ placed at p, so that the whole mass moved by the force which moves P is

$$\frac{P+\frac{1}{4}W+\frac{1}{2}Q+\frac{1}{8}Q}{g}=\frac{8P+2W+5Q}{8g}.$$

To determine now the force or weight which moves this mass, we must find how much of the applied weight P is employed in merely balancing W; this is easily done, because, as W is equally supported by the two branches of rope $p'q$, Cq', one half of W is the portion of P employed in balancing W; hence the moving force is

$$P-\tfrac{1}{2}W,\text{ or }\frac{2P-W}{2},$$

and consequently for the acceleration of P we have

$$F=\frac{2P-W}{2}\div\frac{8P+2W+5Q}{8g}=\frac{8P-4W}{8P+2W+5Q}g$$

$$\therefore v=\frac{8P-4W}{8P+2W+5Q}gt, s=\frac{8P-4W}{8P+2W+5Q}\,\frac{gt^2}{2}.$$

PROBLEM VI.—In a system of pulleys contained in two equal and

separate blocks a single string goes round them all: it is required to determine the acceleration of P fastened at the extremity of the string, and drawing up a weight attached to the lower block.

Let there be altogether n pullies, Q being the weight of each, and let the weight of the lower block, together with its attached load, be W, and let us, as in the preceding problem, ascertain what mass in addition to its own must be incorporated in P to supply the place of the inertia of the system.

The pulley which first received the cord from the ascending weight turns with $\frac{1}{n}$th of the velocity of the pulley which delivers the cord to the power, and therefore, as in the last problem, the mass at P, which will be a substitute for its inertia, is $\frac{1}{2 . n^2} \cdot \frac{Q}{g}$. The circumference of the next pulley in the lower block revolving twice as fast as the former, the mass due to its inertia will be $\frac{4}{2 . n^2} \cdot \frac{Q}{g}$, and so on; hence, for the pulleys, the mass to be substituted will be

$$\frac{Q}{2.n^2 g}(1+2^2+3^2+4^2+\ldots\ldots+n^2) = \frac{Q}{2n^2 g}\left\{\frac{2n^3+3n^2+n}{6}\right\}$$

Again, the velocity of W being $\frac{1}{n}$th of P's velocity, the mass at P, which represents its inertia, will, as in the preceding problem, be $\frac{1}{n^2} \cdot \frac{W}{g}$; hence the inertia of the weights is represented by the mass $\frac{P}{g} + \frac{1}{n^2} \cdot \frac{W}{g}$, and therefore the whole mass moved is

$$\frac{P}{g} + \frac{1}{n^2} \cdot \frac{W}{g} + \frac{Q}{2ng}\left\{\frac{2n^2+3n+1}{6}\right\}.$$

The weight at P which balances W is $\frac{W}{n}$, and consequently that which moves the system is $P - \frac{W}{n} = \frac{nP - W}{n}$, and therefore, dividing this by the mass moved, as above expressed, we have for the acceleration of P; $F = \frac{12n(nP - W)g}{12(n^2 P + W) + Qn(2n^2 + 3n + 1)}$, from which, as in the preceding problem, the expressions for the velocity and space generated in any time t'' are immediately deducible.

Problem VII.—A wheel, whose interior and exteror radii are

r_1, r_2, rolls down an inclined plane (fig. 123), of which the friction is just sufficient to prevent sliding: to determine the circumstances of the motion.

Let i be the inclination of the plane, and F the *effective* accelerative force down it, then, putting w for the mass of the wheel, Fw will represent the effective moving force. But the impressed moving force is wg sin. i, minus the resistance of the friction, which, as it diminishes the amount of the moving force which the body would otherwise have, we may represent by an opposing moving force; let us then call it $w'g$ sin. i. Now if P be the point in contact with the plane at D, where the motion is supposed to have commenced, then, since in any time t'', DP′=P′P, it follows that the rotatory acceleration of P is also F; and, consequently, the acceleration of a particle at the unit of distance from C, that is the angular acceleration, is $\frac{F}{r_2}$; hence wk^2 being the moment of inertia of the wheel round C, $\frac{F}{r_2}wk^2$ will be the actual moment of the system round C. But the only impressed moment is that arising from the friction, it is, therefore, $r_2 w'g$ sin. i; hence, by the principle of D'Alembert, we have, by equating the impressed and effective forces,

$$Fw=(w-w')g\,\text{sin.}\,i;\quad \frac{Fw\,k^2}{r_2}=r_2 w' g\,\text{sin.}\,i;$$

eliminating w', we have $Fw\,r_2^2+Fw\,k^2=wr_2^2\,g$ sin. i

$$\therefore F=\frac{g\,\text{sin.}\,i\,r_2^2}{r_2^2+k^2};$$

which expresses the acceleration of the centre C down the plane, and in which $k^2=\frac{r_1^2+r_2^2}{2}$, (ex. 3, page 188.)

If the cylindrical wheel be indefinitely thin, or when $r_1=r_2$,

$$F=\tfrac{1}{2}\,g\,\text{sin.}\,i.$$

For the time of describing a given space s, we have,

$$s=\tfrac{1}{2}Ft^2,\ t=\sqrt{\left\{\frac{2s(r_2^2+k^2)}{g\,\text{sin.}\,i\,r_2^2}\right\}},$$

and for the velocity acquired, $v=Ft=\sqrt{\left\{\frac{2s\,g\,\text{sin.}\,i\,r_2^2}{r_2^2+k^2}\right\}}$, which expresses also the rotatory velocity of every point on the outer circumference.

To determine the absolute velocity of any proposed point P of the outer circumference in space, or in its cycloidal path, we must compound together these two velocities; so that, calling the absolute velocity V, we have

$$V = 2\,v \cos. \tfrac{1}{2}\,n\,P\,m = 2\,v \times \text{tab. cos. } \tfrac{1}{4} \text{ arc. } P\,p\,m;$$

and the degrees in Ppm are known, since the length $2\,s$ of the arc $PP'm$ is known; therefore the absolute motion of P is determined both as to velocity and direction.

If p be diametrically opposite to the point of contact P′, it appears from this expression for V, that the absolute velocity of any point P at any time varies as the tabular cosine of half the arc Pp; at p this velocity is greatest, being $= 2v$; and at P′ it is least, being for an instant 0, that is, P′ is the centre of spontaneous rotation of the body. The last conclusion, viz. that the point of contact can have no motion along the plane, is an immediate consequence of the conditions of the problem, for if it had any motion along the plane, the body would slide on that point, whereas the friction is supposed to be sufficiently great to prevent sliding down the plane.

If, in consequence of any initial impulse, the rotatory velocity exceed that of translation, P′ will no longer be the centre of spontaneous rotation, but will have a velocity backward greater than that forward, so that the body will move up the plane and will continue to do so till this excess of rotatory velocity is destroyed by the friction; when the body, after being for an instant stationary, will reverse its progressive motion, and roll down the plane with a velocity equal to that of rotation. If the velocity of translation were made to exceed that of rotation the body would partly roll down the plane and partly slide. These deductions are on the supposition that the friction of the plane is just sufficient to prevent sliding when the initial velocity of the body in progression is equal to that in rotation.

Rotatory motion of this kind may be produced by the uncoiling of a thread or riband previously wound about the body; the tension of the thread supplying the place of friction.

PROBLEM VIII.—A sphere, whose mass is P, is placed on the slant side of a smooth prism, whose mass is Q, and a fine thread or riband is fixed to the prism at B (fig. 124,) and coiled 'round the sphere in the plane of its vertical great circle, the object of it being to cause the sphere to roll and not slide down the plane. The base AC of the prism, as well as the horizontal plane on which it is placed, is perfectly smooth, so that it moves along the plane OC, in consequence of the pressure of the rolling sphere. It is required to determine the tension of the string, at any time the pressure on the prism, and the path described by the point of contact P.

In this case the rolling body has two motions, viz. one down the inclined plane, and the other in a horizontal direction, as well as the rotatory motion about its centre. As in last problem, the rota-

tory acceleration of any point in the circumference, must be equal to the acceleration of the point of contact P down the plane; but this rotatory acceleration is wholly produced by the tension T of the thread, it will, therefore, be expressed by dividing this force by the moment of inertia of the sphere, that is, the acceleration of the point of contact is $T \div \frac{2}{5}P$, (p. 190-1.) Let us now deduce another expression for this acceleration of the point P from the actual space BP passed over by it, and for this purpose put $ON = x$, $OA = x_1$, O being the place of A at the commencement, or when P is at B; let, moreover, $BC = a$, $AC = b$, $AB = c$; then the space BP is

$$BP = (x_1 + b - x)\sec. A = (x_1 + b - x)\frac{c}{b},$$

and therefore the second differential coefficient, with respect to the time, must express the acceleration down the plane; that is,

$$\frac{c}{b}\left\{\frac{d^2 x_1}{dt^2} - \frac{d^2 x}{dt^2}\right\} = \frac{5\,T}{2\,P} \quad \ldots . \ (1).$$

The first of these differential coefficients denotes the acceleration of the point A or of the entire prism in a horizontal direction, and the second denotes the acceleration of the point P, or of the sphere in a horizontal direction. Now the motive forces to which these accelerations are due, are equal and opposite, therefore, calling p the pressure on the prism, or $-p$, the resistance against the sphere, we have for the horizontal force on the sphere

$$P\frac{d^2 x}{dt^2} = -p\frac{a}{c} + T\frac{b}{c} \quad \ldots . \ (2),$$

$$\therefore Q\,\frac{d^2 x_1}{dt^2} = p\frac{a}{c} - T\frac{b}{c} \quad \ldots . \ (3).$$

Also, for the vertical force on P, we have

$$P\,\frac{d^2 y}{dt^2} = -Pg + p\frac{b}{c} + T\frac{a}{c} \quad \ldots . \ (4).$$

Consequently, since $y = \frac{a}{b}(x - x_1) \ldots . \ (5)$,

the equations (1) and (4) give

$$\frac{5}{2}\cdot\frac{a}{b}\cdot\frac{b}{c}\,T = Pg - p\,\frac{b}{c} - T\frac{a}{c} \quad \ldots . \ (6).$$

Also the equations (1), (2), (3), give

$$b\frac{5\,T}{2\,P} = \left(\frac{1}{Q} + \frac{1}{P}\right)pa - \left(\frac{1}{Q} + \frac{1}{P}\right)T\,b \quad \ldots . \ (7),$$

and from these two equations we obtain for p and T the values

$$p = \frac{b\,c\,P\,(2\,P + 7\,Q)\,g}{7\,a^2\,(P + Q) + b^2\,(2\,P + 7\,Q)}$$

$$T=\frac{2\,a\,c\,P\,(P+Q)\,g}{7\,a^2(P+Q)+b^2(2\,P+7\,Q)}$$

both of which are constant quantities.

As to the path described by the point of contact P, it is immediately deducible from the conditions of the problem; for, as both bodies commence their motion together, the horizontal momentum of the sphere is equal and opposite to that of the prism, that is,

$$P\frac{dx}{dt}+Q\frac{dx_1}{dt}=0\ \therefore\ P\,x+Q\,x_1=Pb,$$

that is, (equa. 5,)

$$P\,x+Q\,x-Q\frac{b}{a}y=Pb\ \therefore\ y=\frac{a\,(P+Q)}{b\,Q}x-\frac{P\,a}{Q};$$

hence the path is a determinate straight line.

For a very complete and elegant solution to this problem, considered under different modifications, the student may consult a paper by Mr. Mason, in the twentieth number of Leybourn's Mathematical Repository.

CHAPTER V.

ON THE MOTION OF A SYSTEM OF BODIES ACTED ON BY ANY ACCELERATIVE FORCES WHATEVER.

(166.) We propose in this chapter to investigate some very general and remarkable theorems which apply to the motion of a system of bodies acting in any arbitrary manner on each other, and each influenced by any accelerative forces.

Let m, m_1, m_2, &c. represent the masses of the different bodies in the system, x, y, z; x_1, y_1, z_1, &c. the rectangular co-ordinates which mark their position, X, Y, Z, the components of the accelerative forces on m, X_1, Y_1, Z_1, the components of those on m_1, &c. then the motive forces *applied* to any one, as m, will be mX, mY, mZ, and those which actually have place will be $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$; hence the differences of the impressed and effective forces resolved in the directions of the co-ordinates are

$$m\frac{d^2x}{dt^2}-m\,X,\ m\frac{d^2y}{dt^2}-m\,Y,\ m\frac{d^2z}{dt^2}-m\,Z;$$

and, in like manner, for each of the other bodies m_1, m_2, &c. we get similar expressions for the differences between the impressed

and effective forces, and we know, from the principle of D'Alembert (154), that if these differences alone acted on the system it would be kept in equilibrium. But when any forces keep a system in equilibrium these forces must fulfil the conditions (6) and (7), at page 79; hence, in the present case, we must have these two groups of equations, viz.

$$\left.\begin{aligned} \Sigma\left(m\frac{d^2x}{dt^2}\right)&=\Sigma\,(m\mathrm{X}) \\ \Sigma\left(m\frac{d^2y}{dt^2}\right)&=\Sigma\,(m\mathrm{Y}) \\ \Sigma\left(m\frac{d^2z}{dt^2}\right)&=\Sigma\,(m\mathrm{Z}) \end{aligned}\right\}\ldots\ldots(\mathrm{A})$$

$$\left.\begin{aligned} \Sigma\left(m\frac{yd^2x-xd^2y}{dt^2}\right)&=\Sigma\,(my\mathrm{X}-mx\mathrm{Y}) \\ \Sigma\left(m\frac{xd^2z-zd^2x}{dt^2}\right)&=\Sigma\,(mx\mathrm{Z}-mz\mathrm{X}) \\ \Sigma\left(m\frac{zd^2y-yd^2z}{dt^2}\right)&=\Sigma\,(mz\mathrm{Y}-my\mathrm{Z}) \end{aligned}\right\}\ldots\ldots(\mathrm{B}).$$

These two groups of equations, which contain the conditions of the motions of any system of bodies under any circumstances, furnish several general principles of motion; one or two of these we shall proceed to develope.

Let x, y, z, represent the co-ordinates of the centre of gravity of any system of bodies m, m_1, m_2, &c.; then (39) we shall have

$$\mathrm{x}=\frac{\Sigma\,(mx)}{\Sigma\,(m)},\ \mathrm{y}=\frac{\Sigma\,(my)}{\Sigma\,(m)},\ \mathrm{z}=\frac{\Sigma\,(mz)}{\Sigma\,(m)};$$

and taking the second differential co-efficients with respect to t

$$\frac{d^2\mathrm{x}}{dt^2}=\frac{\Sigma\left(m\frac{d^2x}{dt^2}\right)}{\Sigma\,(m)},\ \frac{d^2\mathrm{y}}{dt^2}=\frac{\Sigma\left(m\frac{d^2y}{dt^2}\right)}{\Sigma\,(m)},\ \frac{d^2\mathrm{z}}{dt^2}=\frac{\Sigma\left(m\frac{d^2z}{dt^2}\right)}{\Sigma\,(m)}.$$

Comparing these with the equations (A) we have

$$\frac{d^2\mathrm{x}}{dt^2}=\frac{\Sigma\,(m\mathrm{X})}{\Sigma\,(m)},\ \frac{d^2\mathrm{y}}{dt^2}=\frac{\Sigma\,(m\mathrm{Y})}{\Sigma\,(m)},\ \frac{d^2\mathrm{z}}{dt^2}=\frac{\Sigma\,(m\mathrm{Z})}{\Sigma\,(m)}\ \ldots\ldots(\mathrm{C});$$

or putting M for the whole mass, $m+m_1+m_2+$, &c. of the system, we have

$$\mathrm{M}\frac{d^2\mathrm{x}}{dt^2}=\Sigma\,(m\mathrm{X}),\ \mathrm{M}\frac{d^2\mathrm{y}}{dt^2}=\Sigma\,(m\mathrm{Y}),\ \mathrm{M}\frac{d^2\mathrm{z}}{dt^2}=\Sigma\,(m\mathrm{Z});$$

These equations show that if the whole mass of the system were to be concentrated into the centre of gravity, and this centre to have the same acceleration as in the actual state of the system, the moving force of the whole mass at the centre would be the same

as the entire moving force of the actual system; but Σ (mX) being independent of x, x_1, x_2, &c. is independent of the distances between the several bodies. m, m_1, m_2, &c. and, therefore, would remain the same if these masses were united in a single point, provided only the same accelerative forces were applied parallel to their actual directions; hence the motion which the centre of gravity actually has is the same as it would have if all the system were united there, and the forces applied to it parallel to the directions they really have; whence this general principle of motion, viz.

The centre of gravity of a system of bodies, acted upon by any accelerative forces and mutually influencing each other, moves in space as if the system were united into that centre, and the forces which solicit the bodies were directly applied to it.

(167.) When the system is acted on by no other forces than the mutual attractions of its parts, the second members of the equations (A) must vanish; for, as the action of any two of the bodies is mutual, each will impress on the other the same motive force; and as these forces are opposite to each other, it follows that if the bodies were connected with each other by rigid rods, these, on account of the sequel at opposite pressures at their extremities, would be held in equilibrium, whence all motion in the system would be prevented by the forces applied to its several parts thus connected together; hence these forces must fulfil the six equations of equilibrium, so that we must here have

$$\Sigma\,(mX)=0,\ \Sigma\,(mY)=0,\ \Sigma\,(mZ)=0\ .\ .\ .\ .\ (D)$$

$$\Sigma\,(myX-mxY)=0,\ \Sigma\,(mxZ-mzX)=0,\ \Sigma\,(mzY-myZ)=0\ .\ (E);$$

and, consequently, from the equations (C), we get

$$\frac{d^2\text{x}}{dt^2}=0,\ \frac{d^2\text{x}}{dt^2}=0,\ \frac{d^2\text{z}}{dt^2}=0,\ \text{of which the integrals are}$$

$$\text{x}=a+bt,\ \text{y}=a'+b't,\ \text{z}=a''+b''t$$

where a, b, a', b' a'', b'', are the arbitrary constants, introduced by the integration.

If we eliminate t from any two of these equations there will result a linear equation either between x and y, or between x and z, or else between y and z; it follows, therefore, that the centre of gravity of the system must describe a straight line, and its velocity at any time will be

$$\text{v}=\sqrt{\frac{d\text{x}^2+d\text{y}^2+d\text{z}^2}{dt^2}}=\sqrt{\{b^2+b'^2+b''^2\}};$$

which, being constant, shows that *the motion of the centre of gravity of a system of bodies, whose motions are entirely due to their mutual influences, is both rectilinear and uniform.*

This is called the *general principle of the conservation of the centre of gravity.* If no primitive impulse be given to the centre

of gravity of a system, then b, b', b'', will be each $=0$, and therefore, $v=0$, or the centre will be at rest.

(168.) We have seen at (122,) that the differential expression $ydx - xdy$,* is the differential of double the area described by the projection on the plane of xy of the radius vector of m; hence the sum of the products of each body into the differential expression for the area it traces out on this plane in any time will be expressed by $\frac{1}{2}\Sigma(mydx - mxdy)$. In like manner the corresponding sums for the other planes will be $\frac{1}{2}\Sigma(mxdz - mzdx)$, and $\frac{1}{2}\Sigma(mzdy - mydz)$; or calling these several sums $\frac{1}{2}dS$, $\frac{1}{2}dS'$, $\frac{1}{2}dS''$, and differentiating, we have

$$\Sigma(myd^2x - mxd^2y) = d^2S$$
$$\Sigma(mxd^2z - mzd^2x) = d^2S'$$
$$\Sigma(mzd^2y - myd^2z) = d^2S''$$

and, therefore, from the equations (B) $\Sigma(myX - mxY) = \frac{d^2S}{dt^2}$

$$\Sigma(mxZ - mzX) = \frac{d^2S'}{dt^2}\,;\ \Sigma(mzY - myZ) = \frac{d^2S''}{dt^2}.$$

Now when the system is influenced only by the mutual actions of its parts, we have seen (equa. E,) that the first members of these equations are each $=0$, consequently,

$$\frac{d^2S}{dt^2} = 0\,,\ \frac{d^2S'}{dt^2} = 0\,,\ \frac{d^2S''}{dt^2}\ \therefore\ S = ct\,,\ S' = c't\,,\ S'' = c''t.$$

This result establishes the *general principle of the conservation of areas*, viz. that *in any system of bodies, moving in virtue of their mutual actions on each other, the sum of the products of each body into the projection, on any plane, of the areas, described by its radius vector, is proportional to the time in which those areas are described.*

(169.) We shall investigate one more general principle, which, not being deducible from the general equations (A) and (B), requires that we consider the motion of the system in another point of view.

In every system of bodies, the motion of each is due to the force impressed on it, combined with those which arise from the action of the other parts of the system; the simultaneous action of all these forces determine the motion of any one body m; and, therefore, taking the components of these forces, viz. mX, mY, mZ, where X, Y, Z are the accelerations of the body in the directions of three rectangular axes, we must have, at any time t'', the equations.

* The sectorial area to which this expression applies is measured from the axis of y, as at art. 122; but for the complement of this area, or that measured from the axis of x, the corresponding expression is, of course, $xdy - ydx$. It is easy to see that the law of areas, established in this article, holds in either case.

$$m\frac{d^2x}{dt^2}=mX,\ m\frac{d^2y}{dt^2}=mY,\ m\frac{d^2z}{dt^2}=mZ$$

$$m_1\frac{d^2x_1}{dt^2}=m_1X_1,\ m_1\frac{d^2y_1}{dt^2}=m_1Y_1,\ m_1\frac{d^2z_1}{dt^2}=m_1Z_1,$$

&c. &c. &c.

Multiplying now, the equation in x by

$\frac{2dx}{dt}$, the equation in y by $2\frac{dy}{dt}$, the equation in z by $2\frac{dz}{dt}$;

and, in like manner, the equation in x_1, by $2\frac{dx_1}{dt}$, and so on, and then adding them together and integrating, we shall evidently have the equation

$$m\frac{dx^2+dy^2+dz^2}{dt^2}+m_1\frac{dx^2_1+dy^2_1+dz^2_1}{dt^2}+\&c.=$$

$$2\int m(X\,dx+Y\,dy+Z\,dz)+2\int m_1(X_1\,dx_1+Y_1\,dy_1+Z_1\,dz_1)+\&c.$$

or, in the usual notation,

$$\Sigma\left(m\frac{dx^2+dy^2+dz^2}{dt^2}\right)=2\,\Sigma\int m(X\,dx+Y\,dy+Z\,dz)$$

that is, $\Sigma(mv^2)=2\,\Sigma\int m(X\,dx+Y\,dy+Z\,dz)$ . . . . (F), which is similar to the equation (2) at page 146, when the system consists of but a single point; the second member is also integrable in like circumstances, that is, when X, Y, Z, &c., are functions of x, y, z, &c. and fulfil also the conditions named at page 146.

If these conditions have place, then, as at the page referred to,

$$\Sigma(mv^2)-\Sigma(mv^2_1)=f(x,y,z,x_1,y_1,z_1,\&c.)-$$
$$f(a,b,c,a_1,b_1,c_1,\&c.)$$

the product mv^2 of the mass into the square of its velocity is the *vis viva*, or *living force*, so that when the second member of (F) is an exact differential, we infer that *the sum of the living forces of the system generated in moving from one position to another, depends solely upon the position left and that arrived at, and is independent of the paths which the several bodies have taken, and of the time of describing them.* This is the general *principle of the conservation of living forces.*

This principle always holds, that is to say, the second member of (F) is always an exact differential when the bodies move in virtue of their mutual attractions. To prove this, let r be the distance between two of the bodies of the system m, m_1; then

$$r^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 \ . \ . \ . \ . \ (1);$$

let R be the function of r which expresses the intensity of the attractive force exercised by the unit of m on m_1, and by the unit of m_1 on m; then the whole attractive force of m on m_1 will be Rm; and the whole attractive force of m_1 on m will be Rm_1. The components of the forces Rm will be (see page 147,)

$$Rm\frac{x-x_1}{r},\ Rm\frac{y-y_1}{r},\ Rm\frac{z-z_1}{r},$$

and these, in virtue of (1) above, are the same as

$$Rm\frac{dr}{dx},\ Rm\frac{dr}{dy},\ Rm\frac{dr}{dz}.$$

In like manner, for the components of the force Rm_1, we have

$$Rm_1\frac{dr}{dx_1},\ Rm_1\frac{dr}{dy_1},\ Rm_1\frac{dr}{dz_1}.$$

This last set of components have the same signs as the former set, because the force m_1 R is opposite to m R, and the coefficients $\frac{dr}{dx_1}$, $\frac{dr}{dy_1}$, $\frac{dr}{dz_1}$, obviously involve opposite signs to the coefficients $\frac{dr}{dx}$, $\frac{dr}{dy}$, $\frac{dr}{dz}$.

Substituting the foregoing component forces for X, Y, Z, in the expression $\Sigma\,(m\,\mathrm{X}\,dx+m\,\mathrm{Y}\,dy+m\,\mathrm{Z}\,dz)$, we have

$$m\,m_1\,\mathrm{R}\frac{dr}{dx}\,dx+m\,m_1\,\mathrm{R}\frac{dr}{dy}\,dy+m\,m_1\,\mathrm{R}\frac{dr}{dz}\,dz+m_1\,m\,\mathrm{R}\frac{dr}{dx_1}\,dx_1$$
$$+m_1\,m\,\mathrm{R}\frac{dr}{dy_1}dy_1+m_1\,m\,\mathrm{R}\frac{dr}{dz_1}dz_1=m\,m_1\,\mathrm{R}\,dr,$$

the first side of this equation being mm_1 times the total differential of r, considered as a function of the six variables in equation (1) above. The same reasoning applied to every two bodies in the system, must lead to a similar result, so that,

$$2\,\Sigma\int(m\,\mathrm{X}\,dx+m\mathrm{Y}dy+m\,\mathrm{Z}\,dz)=2\,\Sigma\,.\,m\,m_1\int\mathrm{R}\,dr,$$

which is always integrable, since, by hypothesis, R is a function of r.

CHAPTER VI.

ON THE COMPOSITION OF ROTATORY MOTIONS; AND ON THE PRINCIPAL AXES OF ROTATION OF A SOLID BODY.

On the Composition of Rotatory Motions.

(170.) If a body receive simultaneously impulses which are separately competent to produce rotation about different known axes, the result will be a rotation about a new and determinable axis. It will be sufficient to consider three given axes at right angles to each other, and the problem we propose now to solve is this, viz. when the body tends to turn at the same instant about three rectangular axes, AX, AY, AZ, with the respective angular velocities ω, ω', ω''; to determine the position of the axis about which it actually turns, and the angular velocity of the rotation.

Let m represent any particle of the body (fig. 125,) and let us first consider the motion of it about the axis AY; this will be in a circle pqm, and we shall suppose it in the direction of these letters. The plane of the circle pqm being parallel to the plane of xz, the particle has no motion in the direction of AY, and its motion in the directions of AX, AZ will obviously be the same as if the circle coincided with the plane of xz, we may then, in estimating these motions, suppose such to be the case. Now as m turns towards the plane of xy, its co-ordinate x increases, and its co-ordinate z diminishes; hence the differential of x, with respect to the time, will be positive, that of z negative. The absolute velocity of m about AY will at any time t'', corresponding to the co-ordinates x, y, be in the direction of the tangent mr, taking, therefore, this line to represent it, its components in the directions of AX, AY, will be ms, and ma. As the expression for the absolute velocity is $\mathrm{A}m \,.\, \omega'$ we have $\mathrm{A}m \,.\, \omega'=mr$, and since $ms=mr \sin. mrs=mr \sin. \mathrm{A}=\omega'.\mathrm{A}m \sin. \mathrm{A}=\omega'.mb=\omega' z$, this, therefore, expresses the velocity in the direction of AX which is due to the rotation about AY. In like manner, $ma=mr \cos. rma=mr \cos. \mathrm{A}=\omega' \,.\, \mathrm{A}m \cos. \mathrm{A}=\omega' x$ is the velocity in the direction AZ to be taken negatively; hence the velocities in the directions of AX, AZ, due to the rotation of any particle at the point (x, y, z), at any time t'' is $\omega' z$ and $-\omega' x$.

Consider now the rotation of the particle m about the axis AX (fig. 126,) from Y towards Z, and applying precisely the same reasoning, we get for the velocities in the directions of Z and Y the values ωy and $-\omega z$. And finally, applying the same reasoning when the rotation is about the axis of z from X towards Y (fig. 127), we have, for the velocities in the directions of AY, AX, $\omega'' x$ and $-\omega'' y$.

It follows, therefore, that when all these rotations have place simultaneously, we have, by adding together the above partial velocities along the axes, the following expressions for the whole velocities in those directions, viz.

$$\left.\begin{aligned} \frac{dx}{dt}&=\omega' z-\omega'' y \\ \frac{dy}{dt}&=\omega'' x-\omega z \\ \frac{dz}{dt}&=\omega y-\omega' x \end{aligned}\right\}\ldots\ldots(1).$$

Now we may determine from these expressions about what axis the body actually turns at the instant t'', when the foregoing motions have place simultaneously; for as every particle in the axis of instantaneous rotation is motionless, we must have for all of them

$$\frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{dz}{dt}=0,$$

that is, $\omega' z-\omega'' y=0,\ \omega'' x-\omega z=0,\ \omega y-\omega' x=0,$ and these three equations obviously characterize a straight line in space passing through the origin of the original axes; two of these equations are sufficient to determine the position of this line, as for instance the equations

$$x=\frac{\omega}{\omega''}z,\ y=\frac{\omega'}{\omega''}z\ \ldots\ (1),\ \text{where}\ \frac{\omega}{\omega''}\ \text{and}\ \frac{\omega'}{\omega''}$$

are the trigonometrical tangents of the angles which the projections of the instantaneous axis make with the axis of z; hence if α, β, γ, denote the angles which the instantaneous axis of rotation makes with the axis of x, y, and z, we have (*Anal. Geom.* p. 228,)

$$\cos.\ \alpha=\frac{\omega}{\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}},\ \cos.\ \beta=\frac{\omega'}{\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}}$$

$$\cos.\ \gamma=\frac{\omega''}{\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}},$$

and thus the position of the required axis is determined in terms of the known angular velocities.

To determine the angular velocity of the body about this axis, we need consider only the angular velocity of any single particle chosen at pleasure; let us take a particle on the axis of x; if from it we draw a perpendicular p to the instantaneous axis, then the distance of the particle from the origin being x, we have

$$p=x\ \sin.\ \alpha=x\sqrt{\{1-\cos.^2\}}\ \alpha=\frac{x\sqrt{\{\omega'^2+\omega''^2\}}}{\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}}.$$

Now as for this particle $y=0$, $z=0$, in the second members of (1), we have, for its absolute velocity,

$$V=\sqrt{\frac{dx^2+dy^2+dz^2}{dt^2}}=x\sqrt{\{\omega'^2+\omega''^2\}},$$

and, consequently, for the angular velocity v, we have

$$v=\frac{V}{p}=\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}\ \ldots\ (2),$$

so that the three angular velocities ω, ω', ω'', about three rectangular axes, are equivalent to the singular angular velocity,

$$\sqrt{\{\omega^2+\omega'^2+\omega''^2\}},$$

about an axis inclined to these three, at angles whose cosines are the expressions for cos. α, cos. β, cos. γ, above.

It is obvious, from what has now been said, that when a body revolves about any axis given in position, and with a given angular velocity, we may always resolve the motion into three partial rotatory motions about the three rectangular axes of co-ordinates, these component motions being in the directions assumed at the

outset of this article. For the equations of the axis of rotation compared with the equations (1), and the expression for the angular velocity compared with the expression (2), will furnish three equations among the unknowns $\omega, \omega', \omega''$, which are sufficient to fix their values. Hence, to whatever combination of rotatory motion the rotation which actually has place be due, we may always consider it as the resultant of the three angular motions above considered.

On the principal Axes of Rotation.

(171.) There are some remarkable properties belonging to certain axes of rotation, passing through any point in space, which well deserve notice; for instance, whatever point be chosen, there always exists one axis in reference to which the moment of inertia of the revolving body is a maximum, and another for which the moment of inertia is a minimum; these two axes are always perpendicular to each other, and they, together with a third axis through the same point, perpendicular to them both, are called the *three principal axes of rotation*, passing through that point. When the point chosen is the centre of gravity of the body, these axes have each of them a property peculiar to themselves, which is that neither of them will suffer any pressure from the rotation of the body round it, so that, when this rotation has once commenced, if the axis were to be withdrawn, the rotation would, nevertheless, continue as if it were there.

To establish these interesting properties it will be requisite, first, to obtain a general expression for the momentum of inertia of a revolving body, in reference to any axis whatever.

In order to this, let AB (fig. 128,) be the axis in reference to which the moment is to be determined, and assume a point on it A for the origin of the co-ordinates. Let m be a particle of the body, and of which the co-ordinates are x, y, z; let the perpendicular $m\mathrm{B} = r$, and the distance Am of the particle from the origin δ; also call the angle mAB, ϵ and let α, β, γ; α', β', γ', be the angles which AB, and Am, make with the axes of x, y, z; we shall thus have

$$\delta^2 = x^2 + y^2 + z^2 \; . \; . \; . \; . \; (1)$$

$$\cos. \epsilon = \cos. \alpha \cos. \alpha' + \cos. \beta \cos. \beta' + \cos. \gamma \cos. \gamma';$$

therefore, since $\cos. \alpha' = \dfrac{x}{\delta}$, $\cos. \beta' = \dfrac{y}{\delta}$, $\cos. \gamma' = \dfrac{z}{\delta}$, we have, by substitution, $\delta \cos. \epsilon = x \cos. \alpha + y \cos. \beta + z \cos. \gamma$. (2).

Now in the right-angled triangle mAB we have $r^2 = \delta^2 \sin.^2 \epsilon = \delta^2 - \delta^2 \cos.^2 \epsilon$; therefore, substituting in this value of r^2 the expressions (1) and (2), we have $r^2 = (1 - \cos.^2 \alpha)\, x^2 + (1 + \cos.^2 \beta)\, y^2 + (1 - \cos.^2 \gamma)\, z^2 - 2xy \cos. \alpha \cos. \beta - 2\, xz \cos. \alpha \cos. \gamma -$

$2\,yz \cos.\beta \cos.\gamma$, that is, $r^2 = x^2 \sin.^2 \alpha + y^2 \sin.^2 \beta + z^2 \sin.^2 \gamma - 2\,xy \cos.\alpha \cos.\beta - 2\,xz \cos.\alpha \cos.\gamma - 2\,yz \cos.\beta \cos.\gamma$.

From this equation we immediately get the expression for the moment of inertia $\Sigma(r^2 m)$ of the body, in reference to the axis AB, for putting for brevity

$$\Sigma(x^2m) = \text{A},\ \Sigma(y^2m) = \text{B},\ \Sigma(z^2m) = \text{C}$$
$$\Sigma(xym) = \text{D},\ \Sigma(xzm) = \text{E},\ \Sigma(yzm) = \text{F}$$

the foregoing equation gives

$$\left.\begin{aligned}\Sigma(r^2m) = \text{A} \sin.^2\alpha + \text{B} \sin.^2 \beta + \text{C} \sin.^2\gamma \\ -2\,\text{D} \cos.\alpha \cos.\beta - 2\,\text{E} \cos.\alpha \cos.\gamma - 2\,\text{F} \cos.\beta \cos.\gamma\end{aligned}\right\}(3);$$

which is the general expression for the moment of inertia of the mass M, in reference to any axis AB through the origin of the co-ordinates, and inclined to them at the angles α, β, γ. The position of the rectangular axes of co-ordinates originating at A. being arbitrary, if we could so fix them that they should give $\text{D}=0$, $\text{E}=0$, $\text{F}=0 \ldots . (4)$, the general expression (3) would be considerably simplified, being for such axes reduced to the first line; that is, we should then have,

$$\Sigma(r^2m) = \text{A} \sin.^2\alpha + \text{B} \sin.^2\beta + \text{C} \sin.^2\gamma \ldots . (5).$$

We shall presently show that there really does exist for every point A three rectangular axes, for which the conditions (4) have place, these being what are called the *principal axes of rotation*. But, before we enter upon the proof of this, it will be expedient to show how to transform the equation (3) into another, into which there shall enter, instead of A, B, C, the expressions for the moments of inertia around the three axes chosen for those of the co-ordinates. Now as the distance r' of the particle m from the axis of x is $\sqrt{\{y^2+z^2\}}$, its distance r'' from the axis of y, $\sqrt{\{x^2+z^2\}}$, and its distance r''' from the axis of z, $\sqrt{\{x^2+y^2\}}$, it follows that if we put for the moments of inertia about these axes the symbols A′, B′, C′, we shall have

$$\Sigma(r'^2 m) = \Sigma(y^2+z^2)\,m = \text{B}+\text{C} = \text{A}'$$
$$\Sigma(r''^2 m) = \Sigma(x^2+z^2)\,m = \text{A}+\text{C} = \text{B}'$$
$$\Sigma(r'''^2 m) = \Sigma(x^2+y^2)\,m = \text{A}+\text{B} = \text{C}'.$$

When, therefore, these three moments of inertia are known, the moment, with respect to any other axis AB, will be obtained by substituting in the equation (3) the values of A, B, C, in terms of A′, B′, C′, deduced from these expressions; the result of this substitution is

$$\Sigma(r^2m) = \tfrac{1}{2}\,\text{A}'(\sin.^2\beta + \sin.^2\gamma - \sin.^2\alpha)$$
$$+\tfrac{1}{2}\,\text{B}'(\sin.^2\alpha + \sin.^2\gamma - \sin.^2\beta) + \tfrac{1}{2}\,\text{C}'(\sin.^2\alpha + \sin.^2\beta - \sin.^2\gamma).$$

But since $\cos.^2\alpha + \cos.^2\beta + \cos.^2\gamma = 1 \ldots . (6)$, or $1 - \sin.^2\alpha + 1 - \sin.^2\beta + 1 - \sin.^2\gamma = 1$, or $\sin.^2\alpha + \sin.^2\beta + \sin.^2\gamma = 2$; this equation is the same as $\Sigma(r^2m) = \text{A}'\cos.^2\alpha + \text{B}'\cos.^2\beta + \text{C}'\cos.^2\gamma \ldots . (7)$; which shows that the moment of inertia, with

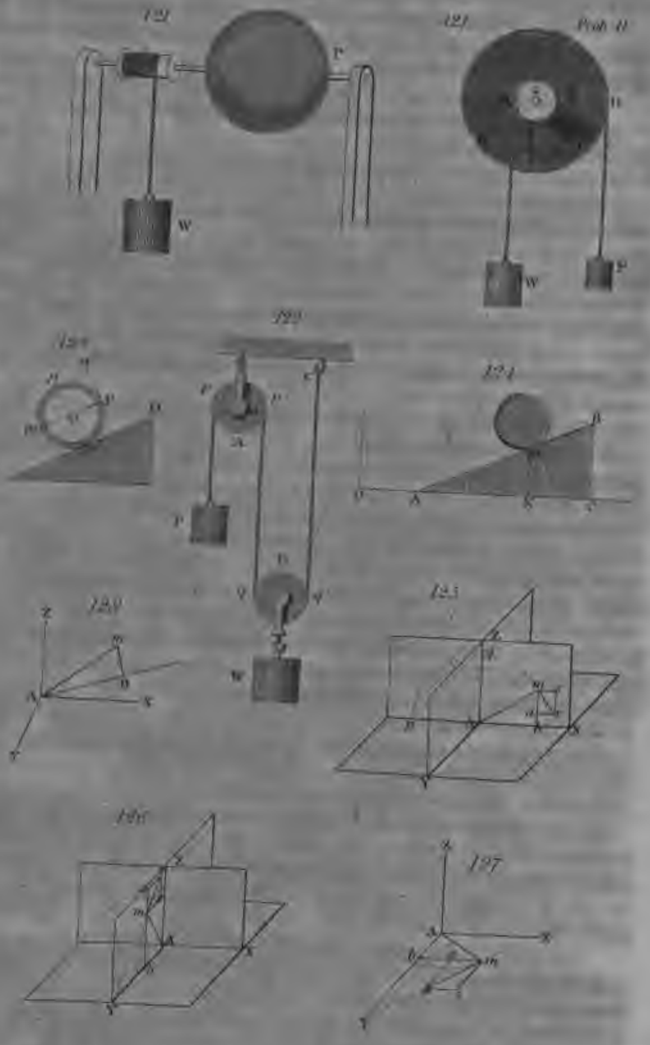
Plate II
120
121
122
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125
126
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128
P
W
W
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W
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c
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q'
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m
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x
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Y
Z

respect to any axis, is equal to the sum of the products, arising from multiplying each of the moments, with respect to the principal axes by the squares of the cosines of the inclinations of the proposed axis to these. On account of the relation (6) between these inclinations, we need introduce but two of them into the expression (7), which may be written

$$\Sigma\,(r^2m)=A'+(B'-A')\cos.^2\beta+(C'-A')\cos.^2\gamma \;.\;.\;.\;.\;(8).$$

The quantities A′, B′, C′, are necessarily positive, being formed from squares multiplied by masses: hence, if A′ is the smallest of the three, every term in this equation will be positive, and, whatever the arbitrary angles β, γ may be, we must always have, in this case, $\Sigma\,(r^2m) > A'$, that is to say, no other line AB through the origin can be found, about which, if the body revolve, the moment of inertia can be so small as when the body revolves about that principal axis which we have taken for the axis of x. But, if A′ is the greatest of the three quantities, then $\Sigma\,(r^2m) < A'$, for every value of β and γ, so that then the principal axis in question will be that for which the moment of inertia is greater than for any other axis through the same origin. A′ may, however, be neither the greatest nor the least of the three moments A′, B′, C′, but may be intermediate between B′, C′; but we may make either of these three quantities stand first in the equation (8), by eliminating from (7) that angle which multiplies the quantity we wish to stand alone, by means of the relation (6), thus if γ be eliminated instead of α, as above, then C′ will stand first, and the same conclusions will then apply to the axis of z instead of to the axis of x; hence, *of the three moments of inertia relative to the principal axes, one of them is a maximum, and another a minimum.*

This conclusion, however, is on the supposition that the principal moments are all unequal; but it may happen that this is not the case; let us suppose then that two of them are equal, as $A'=B'$, then the equation (8) becomes

$$\Sigma\,(r^2m)=A'+(C'-A')\cos.^2\gamma\,;$$

where it is evident that if $A'> C'$, A′ will always be $> \Sigma\,(r^2m)$, provided γ is not 90°, and, with the same condition, A′ will always be $< \Sigma\,(r^2m)$, if $A' < C'$; but, when $\gamma=90$, whatever α and β be, then always $\Sigma\,(r^2m)=A'$; we infer, therefore, that when the principal moments of inertia relative to the axis of x and y are equal, the moments are all equal, for every axis in the plane of xy, and the moment relative to the axis of z, will be a maximum or a minimum, according as $A'<C'$, or $A'>C'$.

If $A'=B'=C'$, then $\Sigma\,(r^2m)=A'$, so that, in this case, all the axes through the origin are principal axes. It follows, therefore, that when more than three principal axes can pass through any point, an infinite can pass through that point.

(172.) It is time now to show that through every point of space three principal axes may always be drawn; that is, three axes in reference to which the equations (4) have place. From the properties just developed we shall, obviously, be led to one or other of these principal axes, if they exist by determining for what values of the arbitrary and independent angles α, β, the general expression (3), for the moment relative to any axis, becomes a maximum or a minimum; so that, for the determination of the position of the axes of greatest or least moment or of the suitable values of α and β, we should have, by the theory of maxima and minima, the two equations $\frac{d\,\Sigma\,(r^2m)}{d\alpha}=0$, $\frac{d\,\Sigma\,(r^2m)}{d\beta}=0$; which are sufficient to fix the values of α and β. The performance of the actual differentiation, here indicated, is an easy matter; but the subsequent elimination of α or of β, in order to obtain a final equation involving only one of these quantities, is a very tedious and troublesome operation, which, at length, conducts to a complicated cubic equation. Instead, therefore, of employing this method of investigation, we shall, after the example of Professor Whewell, adopt the following shorter and more elegant process from Lagrange. The problem is to find the position of three axes of rectangular co-ordinates, x', y', z', such that

$$\Sigma\,(x'\,y'\,m)=0,\ \Sigma\,(x'\,z'\,m)=0,\ \Sigma\,(y'\,z'\,m)=0\ \ldots\ldots\ (\text{A}).$$

Let the three fixed axes, in reference to which the required ones are to be determined, be those of x, y, z, both systems having the same origin.

$$\left.\begin{array}{llr} \text{Let } x' \text{ make with } x, y, z, \text{ angles whose cosines are} & & a,\ b,\ c \\ y' & \ldots\ldots & a',\ b',\ c' \\ z' & \ldots\ldots & a'',\ b'',\ c'' \end{array}\right\}.\ (1);$$

then the angles contained between x and y, between x and z, and between y and z, being right angles, and, consequently, their cosines $=0$, we have (*Anal. Geom.* p. 228, art. 182.)

$$\left.\begin{array}{r} a\,a'+b\,b'+c\,c'=0 \\ a\,a''+b'b''+c\,c''=0 \\ a'a''+b'b''+c'c''=0 \\ \text{and (}\textit{Anal. Geom.}\text{ p. 228, art. 181,)} \quad \\ a^2+b^2+c^2=1 \\ a'^2+b'^2+c'^2=1 \\ a''^2+b''^2+c''^2=1 \end{array}\right\}\ \ldots\ldots\ (2);$$

and these are the six equations of condition, which must be fulfilled by the constants (1). Now we have already seen that the general expression for the moment of inertia, with respect to any axis AB, making the angles α, β, γ, with x, y, z is

$\Sigma (r^2 m) = A \sin.^2 \alpha + B \sin.^2 \beta + C \sin.^2 \gamma.$
$\quad - 2 D \cos. \alpha \cos. \beta - 2 E \cos. \alpha \cos. \gamma - 2 F \cos. \beta \cos. \gamma.$

But if this same axis AB make the angles α', β', γ', with x', y', z', then since, by hypothesis, $\Sigma (x'y'm) = 0$, &c. the moment of inertia, in reference to it will be

$$\Sigma (r^2 m) = A \sin.^2 \alpha' + B' \sin.^2 \beta' + C' \sin.^2 \gamma';$$

where A′, B′, C′, stand for $\Sigma (x'^2 m)$, $\Sigma (y'^2 m,)$ $\Sigma (z'^2 m.)$

These two expressions for $\Sigma (r^2 m)$ are, therefore equal. The latter is the same as

$$A' + B' + C' - A' \cos.^2 \alpha' - B' \cos.^2 \beta' - C' \cos.^2 \gamma',$$

but $A' + B' + C' = \Sigma (x'^2 + y'^2 + z'^2) m = \Sigma (x^2 + y^2 + z^2) m$; and this last expression is what we have before represented by A+B+C; hence, substituting in the first expression, $A - A \cos.^2 \alpha$ for $A \sin.^2 \alpha$, $B - B \cos.^2 \alpha$ for $B \sin.^2 \beta$, and $C - \cos.^2 \gamma$ for $C \sin.^2 \gamma$, and then equating the two, we have

$$A + B + C - (A \cos.^2 \alpha + B \cos.^2 \beta + C \cos.^2 \gamma)$$
$$- (2 D \cos. \alpha \cos. \beta + 2 E \cos. \alpha \cos. \gamma + 2 F \cos. \beta \cos. \gamma)$$
$$= A + B + C - (A' \cos.^2 \alpha' + B' \cos.^2 \beta' + C' \cos.^2 \gamma') \ldots . (3.)$$

Now since (*Anal. Geom.* p. 228, art. 182,)

$$\cos. \alpha' = a \cos. \alpha + b \cos. \beta + c \cos. \gamma$$
$$\cos. \beta' = a' \cos. \alpha + b' \cos. \beta + c' \cos. \gamma$$
$$\cos. \gamma' = a'' \cos. \alpha + b'' \cos. \beta + c'' \cos. \gamma;$$

the last term in (3) becomes

$$A' (a^2 \cos.^2 \alpha + b^2 \cos.^2 \beta + c^2 \cos.^2 \gamma$$
$$+ 2 ab \cos. \alpha \cos. \beta + 2 ac \cos. \alpha \cos. \gamma + 2bc \cos. \beta \cos. \gamma)$$
$$+ B' (a'^2 \cos.^2 \alpha + b'^2 \cos.^2 \beta + c'^2 \cos.^2 \gamma$$
$$+ 2 a' b' \cos. \alpha \cos. \beta + 2 a' c' \cos. \alpha \cos. \gamma + 2 b' c' \cos. \beta \cos. \gamma)$$
$$+ C' (a''^2 \cos. \alpha + b''^2 \cos.^2 \beta + c''^2 \cos.^2 \gamma$$
$$+ 2 a'' b'' \cos. \alpha \cos. \beta + 2 a'' c'' \cos. \alpha \cos. \gamma + 2 b'' c'' \cos. \beta \cos. \gamma);$$

and this expression must be identical with

$$A \cos.^2 \alpha + B \cos.^2 \beta + B \cos.^2 \gamma$$
$$+ 2 D \cos. \alpha \cos. \beta + 2 E \cos. \alpha \cos. \gamma + 2 F \cos. \beta \cos. \gamma;$$

hence, equating the co-efficients of the like terms, we have

$$A'a^2 + B' a'^2 + C'a''^2 = A \ldots . (1')$$
$$A'b^2 + B' b'^2 + C'b''^2 = B \ldots . (2')$$
$$A'c^2 + B' c'^2 + C'c''^2 = C \ldots . (3')$$
$$A'ab + B'a'b' + C'a''b'' = D \ldots . (4')$$
$$A'ac + B'a'c' + C'a''c'' = E \ldots . (5')$$
$$A'bc + B'b'c' + C'b''c'' = F \ldots . (6')$$

These six equations, combined with the six marked (2), are sufficient to determine the twelve unknown quantities which enter them, but we shall only require to determine four of them, viz. a, b, c, and A′, and shall therefore eliminate the rest. In order to this, add together (1′) a, (4′) b, (5′) c, and we have

$$\text{A}'a(a^2+b^2+c^2)+\text{B}'a'(aa'+bb'+cc')+\text{C}\,a''(aa''+bb''+cc'')=\text{A}a+\text{D}b+\text{E}c;$$

or, by the conditions (2), page 226,

$$\left.\begin{array}{ll} & \text{A}'a=\text{A}a+\text{D}b+\text{E}c \\ \text{Similarly } (2')\,b+(6')\,c+(4')\,a \text{ gives} & \text{A}'b=\text{B}b+\text{F}c+\text{D}a \\ \quad\;\; (3')\,c+(5')\,a+(6')\,b \;\ldots\ldots & \text{A}'c=\text{C}c+\text{E}a+\text{F}b \end{array}\right\}\ldots(4).$$

These three equations, together with the condition $a^2+b^2+c^2=1$, are sufficient to determine a, b, c, and A$'$.

By the first two
$$(\text{A}'-\text{A})\,a-\text{D}b-\text{E}c=0$$
$$(\text{A}'-\text{B})\,b-\text{F}c-\text{D}a=0$$
from which, by eliminating c, we have
$$\{(\text{A}'-\text{A})\,\text{F}+\text{ED}\}\,a-\{(\text{A}'-\text{B})\,\text{E}+\text{FD}\}\,b=0$$
$$\therefore b=\frac{(\text{A}'-\text{A})\,\text{F}+\text{ED}}{(\text{A}'-\text{B})\,\text{E}+\text{FD}}\,a\ldots\ldots(5);$$
also eliminating b from the same two equations,
$$\{(\text{A}'-\text{A})(\text{A}-\text{B})-\text{D}^2\}\,a-\{(\text{A}'-\text{B})\,\text{E}+\text{FD}\}\,c=0$$
$$\therefore c=\frac{(\text{A}'-\text{A})\,(\text{A}'-\text{B})-\text{D}^2}{(\text{A}'-\text{B})\,\text{E}+\text{FD}}\,a\ldots\ldots(6).$$

Substituting these values of b and c in the third of equations (4), that is, in $(\text{A}'-\text{C})\,c-\text{E}\,a-\text{F}\,b=0$, there results
$$(\text{A}'-\text{C})\frac{(\text{A}'-\text{A})(\text{A}'-\text{B})-\text{D}^2}{(\text{A}'-\text{B})\,\text{E}+\text{FD}}-\text{E}-\text{F}\frac{(\text{A}'-\text{A})\,\text{F}+\text{ED}}{(\text{A}'-\text{B})\,\text{E}+\text{FD}}=0;$$
$$\text{or, } (\text{A}'-\text{A})\,(\text{A}'-\text{B})\,(\text{A}'-\text{C})-$$
$$\{(\text{A}'-\text{A})\,\text{F}^2+(\text{A}-\text{B})\,\text{E}^2+(\text{A}'-\text{C})\,\text{D}^2\}-2\,\text{FED}=0\ldots\ldots(7),$$
a cubic equation in A$'$. This equation being of the third degree has necessarily, at least, one real root, and consequently, the values of a, b, and c, as determined from the equations (5) and (6) combined with the equation $a^2+b^2+c^2=1$, are real, so that there exists at least one principal axis, viz. the axis of x', and its position is determined by these three equations.

Returning now to the equations (1$'$), (2$'$), &c. and making the same combinations of them as before, only using now a', b', c', instead of a, b, c, we shall, obviously, be led to the same cubic equation (7), except that B$'$ will occupy the place of A$'$; and if we use a'', b'', c'', instead of a, b, c, the resulting cubic will differ from that above only in having C$'$ in place of A$'$. Thus although the first of these cubics determine three positions for the axis of x' (one at least being real); the second, three positions for the axis of y'; and the third, three positions for the axis of z'; yet these systems of threes must coincide, and can, therefore, furnish only three distinct and different directions for the axes of x', y', and z', given by the three roots of (7).

It remains to show that these roots are all real. Suppose two of

them to be impossible, and, therefore, of the form $m+n\sqrt{-1}$, and $m-n\sqrt{-1}$; the quantities a, b, c, are possible, when the root A' is so, and for one of the impossible roots the corresponding quantities a', b', c', will be of the form $p+q\sqrt{-1}$, $p'+q'\sqrt{-1}$, $p''+q''\sqrt{-1}$, and for the other root, a'', b'', c'', will be of the form $p-q\sqrt{-1}$, $p'-q'\sqrt{-1}$, $p''-q''\sqrt{-1}$. Now as

$$a'a''+b'b''+c'c''=0$$

$$\therefore p^2+p'^2+p''^2+q^2+q'^2+q''^2=0,$$

which is absurd, because the sum of a series of squares can never be 0. Hence the three rectangular axes, each having the property (A), really exist through whatever point we require them to be drawn, because the origin may be arbitrary.

(173.) But the existence of the three principal axes may be established in another way, after having shown, as above, that one exists. For suppose the axis of x, whose position is arbitrary, to coincide with this principal axis; then we must have $a=1$, $b=0$, $c=0$, and these values substituted in the three equations (4) in a, b, c, above, give $A'-A=0$, $D=0$, $E=0$, so that the cubic equation (7), from which the values of A' are to be deduced, becomes, by putting these values for D and E,

$$(A'-A)(A'-B)(A'-C)-(A'-A)F^2=0$$

$$\therefore (A'-B)(A-C)-F^2=0,$$

$$\text{or } A'^2-(B+C)A'+BC-F^2=0 \quad \ldots . \; (1).$$

This being a quadratic will furnish two values for A', and to determine the corresponding inclinations a, b, c, we have (4) the two equations $A'b=Bb+Fc+Da$, $a^2+b^2+c^2=1$; but $D=0$, and, since the axis to be determined must make a right angle with that of x, we must have $a=0$, therefore

$$(A'-B)b=Fc\,;\; b^2+c^2=1,$$

so that if θ represent the inclination of the required axis to that of y, then $b=\cos.\theta$, $c=\sin.\theta$, $\therefore \tan.\theta=\frac{c}{b}=\frac{A'-B}{F}$

$$\therefore \tan.2\theta=\frac{2\tan.\theta}{1-\tan.^2\theta}=\frac{2F(A'-B)}{F^2-(A'-B)^2};$$

the denominator of this fraction will be given by adding the identical equation $(C-B)A'-(C-B)B=(A'-B)(C-B)$ to the equation (1), for there results

$$(A'-B)^2-F^2=(A'-B)(C-B) \quad \ldots . \; (2).$$

$$\therefore \tan.2\theta=\frac{2F}{B-C} \quad \ldots . \; (3),$$

and, as the same tangent belongs to two arcs of which the differ ence is 180°, therefore there are two values for θ, of which the difference is 90°, so that besides the principal axis, which has been

made coincident with that of x, there are two others in the plane of zy, inclined to the axis of y in the angles θ and $90+\theta$, or perpendicular to each other.

When we know the position of one of the principal axes, taking it for the axis of x, the position of the other two becomes determinable from the equation (3), just deduced.

(174.) Let us now prove, that if a body revolve about one of its principal axes passing through the centre of gravity, this axis will suffer no pressure from the centrifugal forces of the several particles.

Let the body revolve about the axis of z, then every particle m will describe about this axis the circumference of a circle of radius $\sqrt{\{x^2+y^2\}}$ and, therefore, if ω be the angular velocity of the system, $\omega\sqrt{\{x^2+y^2\}}$ will express the rotatory velocity of any particle m whose co-ordinates are x, y; but the centrifugal force being equal to the square of the velocity, divided by the radius, its general expression here is $\omega^2\sqrt{\{x^2+y^2\}}$ and consequently the strain which any particle m produces on the axis is $m\,\omega^2\sqrt{\{x^2+y^2\}}$; if this force be resolved in directions parallel to x and y, the two components will be $m\omega^2 x$ and $m\omega^2 y$, and the moment of these forces, to turn the body about the axis of y and of x, will be $m\omega^2 xz$ and $m\omega^2 yz$, and therefore, of the forces exercised by all the particles, the moments will be

$$\omega^2\,\Sigma\,(m\,xz) \text{ and } \omega^2\,\Sigma\,(m\,yz)\;\ldots\;(1),$$

if these be each 0, there will be no effort used by the centrifugal forces to *incline* the axis of z towards the plane of xy; such is the case, therefore, when the axis of rotation is a principal axis; hence, in this case, the only effect of the forces $m\omega^2 x$ and $m\omega^2 y$ on the axis, is to move it parallel to itself, or to translate the body in the directions of x and y; the aggregate of these forces is

$$\omega^2\,\Sigma\,(m\,x) \text{ and } \omega^2\,\Sigma\,(m\,y)\;\ldots\;(2),$$

and if these be each 0, the forces will use no effort to move the body or to press the axis: and they are 0 when the axis passes through the centre of gravity; we conclude, therefore, that when a body revolves about one of its principal axes passing through the centre of gravity, the rotation causes no pressure whatever upon the axis, which may, therefore, be removed without at all affecting the motion of the body, the rotation once impressed continuing permanent. On this account the principal axes through the centre of gravity are called the *axes of permanent rotation*, or by some, *the natural axes of rotation*.

(175.) If the initial rotatory motion of the body be not about a permanent axis of rotation, the effects of the centrifugal forces on the axes cannot be destroyed, inasmuch as the foregoing conditions cannot obtain; these forces, therefore, will alter the axis of rotation, and the body will at every instant of its motion, if free, turn about

a different axis, called the *instantaneous axis of rotation*; and it may be proved that if this axis does not at the commencement of motion coincide with a permanent axis, it can never coincide with one afterwards, so that whenever we observe a body to revolve about one axis during any time, however short, we may conclude that it has continued to revolve about that axis from the commencement of the motion, and that it will continue to revolve about it for ever, unless checked by some extraneous obstacle.

These particulars will be more completely established in the following articles.

We may further remark here, that when a body revolves about any one of the principal axes passing through the fixed point, which is taken for the origin, although this point be not the centre of gravity of the body, yet the expressions (1) will still be 0, so that the revolving body will use no effort to cause the axis to turn about the origin in any direction; but as the expressions (2) will not be 0, the axis must sustain a pressure in the directions of x of y, which would cause a tendency in the axis of z to turn about those of y and x, unless these pressures were wholly exerted upon the fixed point or origin; that is, unless the resultant of all the pressures passed through this point; such, therefore, in virtue of the former conditions, must be the case; so that through any given fixed point in a body, there may always be drawn three axes around which the body may turn uniformly without changing its original axis of rotation, although it would be at liberty to do so, as it is free to move in any direction about the fixed point. In order, therefore, that a body retained at rest by a single fixed point may, by means of an impulse, receive a permanent motion of rotation, it is necessary and sufficient that the impulse be such as to cause a percussion on one of the principal axes of the body, through the point, equivalent to a single force applied to this point perpendicularly to the same axis.

It remains now for us to prove the assertion above, viz. that the instantaneous axis of rotation can at no instant coincide with a permanent axis unless the body has continued to revolve about this axis from the commencement of the motion, and in order to this it will be convenient, first, to ascertain the equations which express the general theory of a body's rotation about its centre of gravity, and then to discuss those particular forms of them which arise from supposing the rotation to take place about the principal axes.

(176.) The group of equations (A, B,) at art. (166), which expresses the conditions of the motions of any system of bodies mutually connected, and each acted upon by any accelerative forces, obviously holds when the system constitutes a solid body; we may regard them, therefore, as embodying the analytical theory of the

motion of a solid body, of which each particle is acted upon by any accelerative forces.

The first three of the equations referred to, completely determine the progressive motion of the body in space, or the path described by its centre of gravity, furnishing for this purpose the requisite equations

$$\frac{d^2x}{dt^2}=\frac{\Sigma(mX)}{M},\quad \frac{d^2y}{dt^2}=\frac{\Sigma(mY)}{M},\quad \frac{d^2z}{dt^2}=\frac{\Sigma(mZ)}{M},$$

where the sign Σ includes under it all the particles m, m_1, m_2, &c. of the mass M, which are acted upon by the accelerative forces X, X_1, X_2; Y, Y_1, Y_2, &c.

The remaining three equations (B) must be those which determine the *rotatory* motion of the body round this moving centre; or if the centre of gravity remain fixed, and the body be free to move round it in every direction, then the three equations (B) must be sufficient to determine the circumstances of the rotatory motion arising from the action of the same accelerative forces.

The rotatory motion thus produced, on the supposition that the centre of gravity is fixed, must, since the progressive and rotatory motions are independent of each other, be really that which accompanies the progressive motion the body actually has when this centre is free, and which progressive motion is that which the centre would have if all the accelerative forces acted upon it as a single free point, so that the absolute motion in space of any particle of the body is compounded of these two.

Supposing then the centre of gravity of the body to be fixed, if we place there the origin of the co-ordinates, all that concerns the rotatory motions of the body will be comprised in the equations

$$\Sigma\left(\frac{y\,d^2x}{dt^2}m-\frac{x\,d^2y}{dt^2}m\right)=\Sigma\,(Xy\,m-Y\,x\,m)$$

$$\Sigma\left(\frac{x\,d^2z}{dt^2}m-\frac{z\,d^2x}{dt^2}m\right)=\Sigma\,(Z\,x\,m-X\,z\,m)$$

$$\Sigma\left(\frac{z\,d^2y}{dt^2}m-\frac{y\,d^2z}{dt^2}m\right)=\Sigma\,(Y\,z\,m-Z\,y\,m).$$

Let us represent, as at (170) the angular velocities of the body about the three axes of co-ordinates by ω, ω', ω'', then, instead of the coefficients $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, we may introduce their values from equation (1), page 221, so that the first of the preceding equations will be, by transposing,

$$\Sigma\,(Y\,x\,m-X\,y\,m)=\Sigma\,x\frac{d\,(\omega''\,x-\omega z)}{dt}m-\Sigma\,y\frac{d\,(\omega' z-\omega'' y)}{dt}m.$$

If we actually perform the differentiation here indicated, and always substitute for $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, their values as given by the equations referred to, we shall have

$$\Sigma(\text{Y}\,x\,m-\text{X}\,y\,m)=\frac{d\omega''}{dt}\Sigma(x^2+y^2)m+\omega\omega'\,\Sigma(x^2+y^2)m+$$

$$(2\omega''^2+\omega'^2-\omega^2)\,\Sigma\,(xym)$$

$$-(\omega''\omega+\frac{d\omega'}{dt})\,\Sigma\,(yz\,m)-(\omega''\,\omega'+\frac{d\omega}{dt})\,\Sigma\,(\,x\,z\,m)+$$

$$\omega\,\omega'\,\Sigma\,(x^2-y^2)\,m.$$

Now let us suppose the axes of co-ordinates to be the principal axes of the body, then we know that

$$\Sigma\,(xy\,m)=0,\ (xz\,m)=0,\ (yz\,m)=0;$$

hence, putting

$$\Sigma\,(y^2+z^2)\,m=\text{A},\ \Sigma(x^2+z^2)\,m=\text{B},\ \Sigma(x^2+y^2)\,m=\text{C}$$

$$\therefore\ \Sigma(x^2-y^2)\,m=\text{B}-\text{A},$$

the foregoing equation becomes simply

$$\Sigma\,(\text{Y}\,x\,m-\text{X}y\,m)=\text{C}\frac{d\omega''}{dt}+(\text{B}-\text{A})\,\omega\,\omega',$$

and, in a precisely similar way, we obtain

$$\Sigma\,(\text{X}\,z\,m-\text{Z}\,x\,m)=\text{B}\frac{d\omega'}{dt}+(\text{A}-\text{C})\omega\omega''$$

$$\Sigma\,(\text{Z}\,y\,m-\text{Y}\,z\,m)=\text{A}\frac{d\omega}{dt}+(\text{C}-\text{B})\,\omega'\,\omega''.$$

(177.) Suppose now that no accelerative forces, X, Y, &c. solicit the body, these equations become in that case

$$\left.\begin{aligned}\text{A}\,\frac{d\omega}{dt}+(\text{C}-\text{B})\,\omega'\,\omega''=0\\ \text{B}\,\frac{d\omega'}{dt}+(\text{A}-\text{C})\,\omega\,\omega''=0\\ \text{C}\,\frac{d\omega''}{dt}+(\text{B}-\text{A})\,\omega\,\omega'=0\end{aligned}\right\}\ldots.\ (1).$$

In these equations A, B, C, are constant for the same body, and putting for abridgment

$$\frac{\text{B}-\text{C}}{\text{A}}=\text{L},\ \frac{\text{C}-\text{A}}{\text{B}}=\text{M},\ \frac{\text{A}-\text{B}}{\text{C}}=\text{N};$$

they become $d\omega=\text{L}\,\omega'\,\omega''\,dt$, $d\omega'=\text{M}\,\omega\,\omega''\,dt$, $d\omega''=\text{N}\omega\,\omega'\,dt$. Multiplying these severally by ω, ω', ω'', and putting $\omega\omega'\omega''dt=d\phi$, we have $\omega d\,\omega=\text{L}\,d\phi$, $\omega'd\omega'=\text{M}\,d\phi$, $\omega''d\omega''=\text{N}\,d\phi$, and the integrals of these are

$$\omega^2 = 2L\phi + a^2,\ \omega'^2 = 2M\phi + b^2,\ \omega''^2 = 2N\phi + c^2 \ldots\ldots (2),$$

where a, b, c, are what ω, ω', ω'' become when t, which is the independent variable in the function ϕ, becomes 0; that is, those constants are the initial rotatory velocities about the axes.

Consequently, substituting these values for $\omega\omega'\omega''$ in the equation $dt = \frac{d\phi}{\omega\omega'\omega''}$, we shall have, for the determination of ϕ, corresponding to any time t'', in functions of t, the differential equation

$$dt = \frac{d\phi}{\sqrt{\{(2L\phi+a^2)(2M\phi+b^2)(2N\phi+c^2)\}}}.$$

Suppose now that an initial velocity a is given to the body about one only of the principal axes, then $b = 0$, $c = 0$, and this expression becomes $dt = \frac{1}{2\sqrt{\{MN\}}} \cdot \frac{d\phi}{\phi\sqrt{\{2L\phi+a^2\}}}$, that is, replacing

$2L\phi + a^2$ by its value ω^2, and $d\phi$ by its value $\frac{\omega d\omega}{L}$,

$$dt = \frac{1}{\sqrt{\{MN\}}} \cdot \frac{d\omega}{\omega^2 - a^2},$$

and the integral of this is

$$C + t\sqrt{\{MN\}} = \frac{1}{2a}\log.\frac{\omega - a}{\omega + a}$$

$$\therefore e^{2aC}.\, e^{2at\sqrt{MN}} = \frac{\omega - a}{\omega + a} \ldots\ldots (3).$$

Now the constant C must be determined so that when $t=0$, ω may be $=a$, that is, the first member must vanish for $t=0$; hence e^{2aC} must $=0$, or $C = -\infty$, consequently, there must always be $\omega = a$, and therefore (2) always $\phi=0$, $\omega'=0$, $\omega''=0$; consequently, as before shown, *the impressed velocity about one of the principal axes of rotation continues perpetual and uniform.*

We see, moreover, from the equation (3), that *if the instantaneous axis of rotation does not coincide with a principal axis at the commencement of motion, it can never afterwards coincide with it:* for if we suppose the coincidence to take place at any epoch, and that the angular velocity is then a, then, measuring t'' from that epoch, the foregoing equation must give $\omega = a$ when $t=0$, which requires, as shown above, that $C = -\infty$, and therefore ω must be always $=a$, for every value, positive or negative, of t.

(178.) Let us now see what must be the conditions, in order that the instantaneous axis of rotation, if it do not accurately coincide with one of the principal axes, may yet always be very nearly coincident.

Let us suppose the axis of instantaneous rotation to be nearly co-

incident with the axis of z; then considering the angular motion to be the resultant of three, about the three principal axes, the velocities ω, ω', about the axes of x and y, are, by hypothesis, to be very small in comparison with the angular velocity ω'' about the axis of z, because the body turns almost entirely about this axis. The expression for the sine of γ, the angle which the instantaneous axis makes with that of z, is (p. 222,)

$$\sin.\gamma = \frac{\sqrt{\{\omega^2+\omega'^2\}}}{\sqrt{\{\omega^2+\omega'^2+\omega''^2\}}}.$$

Now, on account of the smallness of both ω and ω', the third of the equations (1) p. 233, becomes $C\dfrac{d\omega''}{dt}=0$ very nearly; so that ω'', the velocity round the axis of z, is very nearly constant. Call it $\omega''=n$, then the remaining equations of (1) become

$$\left.\begin{array}{l} A\dfrac{d\omega}{dt}+(C-B)n\omega'=0 \\ B\dfrac{d\omega'}{dt}+(A-C)n\omega=0 \end{array}\right\}\dots\dots(1).$$

By differentiating the first of these, we have

$$A\frac{d^2\omega}{dt^2}+(C-B)n\frac{d\omega'}{dt};$$

but, from the second, $\dfrac{d\omega'}{dt}=\dfrac{C-A}{B}n\omega$; hence, by substitution,

$$\frac{d^2\omega}{dt^2}+\frac{(A-C)(B-C)}{AB}n^2\omega=0$$

or, putting for brevity the coefficient of $\omega=l^2$,

$$\frac{d^2\omega}{dt^2}+l^2\omega=0\dots\dots(2).$$

The integral of this equation is, (see Int. Calc. p. 233,)

$$t+c'=\frac{1}{l}\sin.^{-1}\frac{\omega}{c}.$$

If at the commencement of motion, or when $t=0$, ω were accurately 0, the constant c' would necessarily be 0; as it is, however, c' must be very small: calling lc', k, we have

$$\omega=c\sin.(lt+k)\ \therefore\ \frac{d\omega}{dt}=lc.\cos.(lt+k);$$

and, substituting this last expression in the first of (1), we get for ω' the value $\omega'=\dfrac{Alc.\cos.(lt+k)}{n(B-C)}$.

Here also we may observe that if, at the commencement of the motion, the instantaneous axis accurately coincided with the axis of z, c would necessarily be 0, for otherwise ω and ω' would never

be both 0; so that we again infer what has been otherwise proved, viz. that ω and ω' if 0 at the beginning are always 0. But, if, as we here suppose, ω and ω' are not accurately 0 at the commencement, then c, instead of being 0, must be very small; consequently, if l involves no impossible quantity, ω and ω' must always be small, however great t may be, for be this as great as it may the factors sin. $(lt+k)$, cos. $(lt+k)$, can never exceed unity.

The expression for l is $l=n\sqrt{\left\{\frac{(A-C)(B-C)}{AB}\right\}}$; where A, B, C are essentially positive; hence that the expression may be possible A—C, B—C, must be either both positive or both negative; that is, C, the moment of inertia with respect to that axis about which the axis of instantaneous rotation perpetually oscillates, must be either the least or the greatest of the three moments A, B, C.

If we were to integrate (2) on the hypothesis that l is imaginary, or l^2 negative, the resulting expressions for ω, ω', would be exponentials, t entering as an exponent; their values would, therefore, increase continually with t, showing that the supposition of these quantities continuing small, after the commencement, is inadmissible.

We conclude, therefore, that when a body commences to revolve about a principal axis, it will perpetually do so: when it happens that any extraneous cause deranges this uniform rotation a little, the body will, nevertheless, always have the new axes of rotation, which it perpetually turns about very near to the original axis, provided this happened to be either the axis of greatest or of least moment; but if it happened to be the axis of mean moment, then, however trifling the derangement may have been, its effects will increase with the time, and the body will depart altogether from its original motion. We hence say that the rotation about the principal axis of greatest or of least moment is *stable*, while the rotation about the axis of mean moment is *unstable*.

CHAPTER VII.

MISCELLANEOUS DYNAMICAL PROBLEMS.

PROBLEM I.—(179.) IF a body revolve about any centre of force S, (fig. 129,) and if the velocity at an apse A is v, the expression for the sector ASP, described by the radius vector in any time t from A, will be ASP$=\frac{1}{2}$ AS . v . t, required the proof.

Taking S for the origin of the rectangular axes, and SA for the

axis of x, the components of the velocity in the curve are always $\frac{dx}{dt}$ and $\frac{dy}{dt}$; at the apse, the velocity in the curve being in a parallel direction to the axis of y, and being 0 in the direction of x, we have, at this point, $\frac{dx}{dt}=0$, $\frac{dy}{dt}=v$; therefore, since (122) $\frac{xdy-ydx}{dt}=c$, we must have at A, $xv=\mathrm{AS}\,.\,v=c\ \therefore\ \frac{1}{2}ct=\text{sector ASP}=\frac{1}{2}\,\mathrm{AS}\,.\,v\,.\,t$.

Problem II.—To determine the curve, such that if it revolve about its axis placed vertically, with a given angular velocity, a heavy ring at liberty to slide along it shall remain wherever it is placed, (fig. 130.)

Let CBA be the curve, and ω the given angular velocity, and let ω be the point where the ring is placed: draw RN a normal to the curve and RM perpendicular to the axis; also let ST be a tangent at R, and call the angle NRM, &c. The absolute velocity of R is $\mathrm{MR}\,.\,\omega$, and, therefore, the centrifugal force is $\frac{\mathrm{MR}^2\,.\,\omega^2}{\mathrm{MR}}=\mathrm{MR}\,.\,\omega^2$, in the direction RR′; the component of this, in the direction RT of the curve, is $\mathrm{MR}\,.\,\omega^2 \sin.\,\alpha$, and the component of the force g of gravity in opposition to this, that is, in the direction RS, is $g\cos.\,\alpha$;

hence, in order that the ring may rest, these two forces must balance each other, $\therefore\ \mathrm{MR}\,.\,\omega^2 \sin.\,\alpha=g\cos.\,\alpha$

$$\therefore\ \mathrm{MR}\,.\,\omega^2\frac{\mathrm{MN}}{\mathrm{MR}}=\omega^2\,.\,\mathrm{MN}=g\ \therefore\ \mathrm{MN}=\frac{g}{\omega^2};$$

that is, the subnormal is constant, and, hence, the curve is a parabola with its vertex downward. Upon similar principles we may determine the concave surface which a fluid presents when a rotatory motion is given to the vessel which contains it. This surface is a paraboloid.

Problem III.—An upright cylinder standing on a smooth horizontal plane has a string coiled several times round it in the plane of its base; one end is fixed to the cylinder, and to the other is attached a body P, to which a velocity is given in the direction of the string; to determine the motion, (fig. 131.)

Let v be the progressive velocity of the cylinder at any instant, then M being its mass, $\mathrm{M}v$ will be the impulsive force due to this velocity; as it acts at the extremity of the radius r of the base, and in the direction of a tangent, it will communicate to the circumference of the cylinder the rotatory velocity $\frac{\mathrm{M}v\cdot r^2}{\mathrm{M}k^2}=2v$; therefore, the absolute velocity of every point of the unwound string must, at

that instant, be $3v$, which, therefore, expresses the corresponding velocity of P; hence, at that instant, the whole momentum of the system is $Mv+3Pv$; but if the given velocity originally communicated to P be v_1, the momentum communicated will be Pv_1, $\therefore Mv+3Pv=Pv_1 \therefore v=\frac{Pv_1}{M+3P}$; this, therefore, is the velocity of the cylinder in progression, at any instant, and twice this is the rotatory velocity of the circumference; the velocity is, therefore, uniform, so that the motion of the cylinder is wholly due to the initial impulse it receives from P; P, therefore, never afterwards acts on the cylinder.

PROBLEM IV.—A uniform straight rod AB (fig. 132,) is placed in an assigned position upon a smooth horizontal plane, and one end of it, B, is drawn uniformly along the straight line CD with a given velocity v; it is required to find the position of the rod at any time, and its angular velocity. See note at page 257.

Let G be the centre of gravity of the rod, then the uniform motion of B along the straight line CD may be considered as the consequence of an impulse at G, in the direction GD′, parallel to CD. As the progressive velocity thus generated is v, the value of the impulsive force is $2a \cdot v$, a representing the mass of half the rod or of AG. But, as this force really acts at B instead of G, there would be generated in addition to the progressive velocity a uniform angular velocity about G, if B were not constrained to continue on the line CD; as it is the angular motion must be about B. Now the same force which applied at B produces any angular motion of the rod about G, would, obviously, if applied at A, and in an opposite direction, produce the same angular motion; whatever angular motion, therefore, has place in the present case, is due to the force $2\,av$ applied, at the commencement of motion, to the point A in the direction AC′. Calling, therefore, the angle ABY, which the rod makes with the perpendicular BY at the beginning ω_1, we have, for the constant angular velocity about B,

$$\frac{d\omega}{dt}=\frac{2\,av\times BE}{\frac{1}{3}(2\,a)^2}=\frac{2\,av\times 2\,a\cos.\omega_1}{\frac{4}{3}a^2}=\frac{3}{4}\cdot\frac{v}{a}\cos.\omega_1;$$

hence, at any time t'', the rod will make with BY an angle ω equal to $\frac{3}{4}\,t\cdot\frac{v}{a}\cos.\omega_1$, and the length of path gone over by B will have been equal to tv, so that the position and angular velocity of the rod at any time is completely determined.

The curve traced out by any point in the moving rod is, obviously, a species of cycloid, for each point describes uniformly the circumference of a circle, whose centre B is uniformly moving

along a straight line; that point in the rod whose rotatory velocity is equal to the progressive velocity of B will describe the common cycloid; for if with centre B a circle be described through this point, and then a tangent to it be drawn parallel to CD, this circle, by rolling on the tangent so that its centre, or the point of contact, may move with the proposed velocity of B, will obviously cause every point in the circumference to revolve with that same velocity, and thus the point in question will trace the path which it actually has, and which must, therefore, be the common cycloid.

Problem V.—Suppose a heavy particle is placed at a given point in a perfectly smooth narrow tube of a given length, and suppose the tube to be whirled about one end as a centre with a given angular velocity in a horizontal plane; it is required to determine the velocity and direction of the particle when it leaves the tube; the motion being solely generated by the revolving tube, (fig. 133.)

Let SA$=a$ be the tube in its first position, and B the place of the particle. Call SB, b, and the distance SP of the particle at any time t'', r; let also v represent the uniform angular velocity of the tube, and SC its position, when the particle quits it.

As no centripetal force acts on the particle, its motion along the tube is entirely due to the centrifugal force rv^2 (art. 138), that is $\frac{d^2r}{dt^2}=rv^2$.

Multiplying this by $2\,dr$, and integrating, we have

$$\frac{dr^2}{dt^2}=r^2v^2+\text{C, also } 0=bv^2+\text{C} \therefore \frac{dr^2}{dt^2}=v^2(r^2-b^2)\ .\ (1);$$

hence, when $r=a$, or when the body arrives at the mouth C of the tube, its velocity in the direction CE of the tube is $v\sqrt{\{a^2-b^2\}}$ also its velocity in the direction CD, perpendicular to this is va; hence its velocity in its path at that point is the component of these, viz. $v\sqrt{\{2\,a^2-b^2\}}$ and for the angle ECF, which the direction makes with the tube, we have

$$\tan.\angle\,\text{ECF}=\frac{\text{CD}}{\text{CE}}=\frac{va}{v\sqrt{\{a^2-b^2\}}}=\frac{a}{\sqrt{\{a^2-b^2\}}}.$$

This angle, added to the angle S, will give the position of CF, with respect to AS. If T represent the time in which the tube passes from the position SA to SC, then since v is the angle described in one second, Tv will express the angle S, and to find T, we have, from the equation (1),

$$t=\frac{1}{v}\log.\frac{r+\sqrt{\{r^2-b^2\}}}{b}\ \therefore \text{T}=\frac{1}{v}\times\log.\frac{a+\sqrt{\{a^2-b^2\}}}{b};$$

so that the angle S is expressed by $\log.\frac{a+\sqrt{\{a^2-b^2\}}}{b}$.

$$m\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy=g\,(mdx-dy)\ \ldots\ldots (3);$$

and, by integration, $\frac{1}{2}m\frac{dx^2}{dt^2}+\frac{1}{2}\frac{dy^2}{dt^2}=g\ (mx-y)+c\ \ldots\ldots (3)$;

or, since in virtue of the condition (2), $\frac{dy^2}{dt^2}=\frac{x^2}{y^2}\cdot\frac{dx^2}{dt^2}$ this equation gives

$$\frac{dx^2}{dt^2}=y^2\,\frac{2\,g\,(mx-y)+2c}{x^2+my^2};$$

which expresses the square of the velocity of W. The determination of c depends upon the initial position, when the velocity is 0; thus $c=g\ (y_1-mx_1)$, where x_1 and y_1 are the values of x and y, at the commencement of motion.

To find when the velocity is again 0 we have from (3), by introducing this value of c, the equation $g\,(mx-y)+g\ (y_1-mx_1)=0$; or, substituting for y its value $\sqrt{\{a^2+x^2\}}$, and reducing, we have the quadratic equation

$$(1-m^2)\,x^2-(2\,my_1-2\,m^2x_1)\,x-(y_1-mx_1)^2+a^2=0.$$

One of the roots of this equation is by hypothesis, $x=x_1$, and, if we represent the other by x_2, we have, by the theory of equations,

$$x_1+x_2=\frac{2m\,(y_1-mx_1)}{1-m^2}.\quad\therefore x_2=\frac{2\,m\,y_1-(m^2+1)\,x_1}{1-m_2}.$$

If $m=1$, that is, if W=2P, W will continually descend, and never become stationary, as remarked at page 30, and the same will of course happen if W > 2P; hence, for the system to become stationary at any time after the commencement of motion, W must be less than 2P, and the distance below the horizontal line, at which this will take place, will be given by the above value of x_2, so that the weight W will move continually backward and forward between the two points x_1, x_2. Whenever the relation between 2P and W is such as to render x_2 negative, it will be impossible for the system to become stationary at any time after the commencement.

This problem may be solved otherwise as follows:

Since the components of the forces P, P′ in nC, nC′ are together equal to $2\,\mathrm{P}\,\frac{x}{y}$, a portion of W, equal to this, is expended in balancing the equal weights P, P′, hence the *moving* force is only $\mathrm{W}-2\,\mathrm{P}\,\frac{x}{y}$, and this, divided by the inertia opposed to motion at the point n, must give the acceleration $\frac{d^2x}{dt^2}$ of W. The inertia due to W is its own mass $\frac{\mathrm{W}}{g}$; the inertia of the other bodies must be expressed by such a mass placed at n, or incorporated with W, as

when multiplied by the acceleration of W will give the same motive force in the vertical direction that the real bodies P, P′, acting along nC, nC' have in the vertical direction. The motive force of $P+P'$, or $2P$, in the direction nW is $2\frac{P}{g}\cdot\frac{x}{y}\cdot\frac{d^2y}{dt^2}$; to determine therefore what mass M, having the acceleration $\frac{d^2x}{dt^2}$, must have the same motive force, we have

$$M\frac{d^2x}{dt^2}=2\frac{P}{g}\cdot\frac{x}{y}\cdot\frac{d^2y}{dt^2}\ \therefore\ M=2\frac{P}{g}\cdot\frac{x}{y}\left(\frac{d^2y}{dt^2}+\frac{d^2x}{dt^2}\right).$$

Dividing, therefore, the motive force $W-2P\frac{x}{y}$ by the whole inertia at n, we have for the acceleration $\frac{d^2x}{dt^2}$ the equation

$$\frac{g\left(W-2P\frac{x}{y}\right)}{W+2P\frac{x}{y}\left(\frac{d^2y}{dt^2}\div\frac{d^2x}{dt^2}\right)}=\frac{d^2x}{dt^2},$$

therefore, putting as before

$$m \text{ for } \frac{W}{2p}, \text{ and } \frac{dy}{dx} \text{ for } \frac{x}{y},\ g(m\,dx-dy)=m\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}dy,$$

which is the same as equation (3) before determined.

Problem VIII.—Two given bodies P, Q (fig. 136,) are connected together by a string, which passes over a fixed pulley at a given distance from a smooth horizontal plane. It is required to determine the circumstances of the motion when P is drawn along the plane by the descending weight Q.

Let the velocity of Q be v, and the velocity of P, u; then, calling PB, x, and PC, s, CQ, y, and the angle P, α, we have

$$\frac{v}{u}=\frac{ds}{dx}=\cos.\alpha=\frac{x}{s};$$

hence the actual motive force of P, in the direction PC, and of which the force in PB is the component, is

$$\frac{du}{dt}\cdot\frac{P}{g}\cdot\frac{1}{\cos.\alpha}=\frac{du}{dt}\cdot\frac{P}{g}\cdot\frac{u}{v}\ \ldots\ (1),$$

the impressed force on P is gravity and the resistance of the plane so that no moving force is impressed.

Again, the actual force of Q is $\frac{dv}{dt}\cdot\frac{Q}{g}$, and the impressed force is the weight Q. Hence, by D'Alembert's principle, the force (1,) acting in the direction CP, equilibrates the force

$$Q - \frac{dv}{dt} \cdot \frac{Q}{g} \dots (2),$$

acting in the direction CB, that is,

$$\frac{du}{dt} \cdot \frac{u}{v} \cdot P = Q\,g - \frac{dv}{dt} \cdot Q$$

$$\therefore P\,u\,du + Q\,v\,dv = Q\,g\,v\,dt = Q\,g\,dy$$

and integrating this equation, $P\,u^2 + Q\,v^2 = 2\,Q\,g\,(y - c)$, c being the value of y at the commencement of motion, therefore, since

$$u^2 = v^2 \frac{s^2}{x^2},$$

we have $v^2 = \dfrac{2\,Q\,g\,x^2\,(y - c)}{P\,s^2 + Q\,x^2}$ for the square of the velocity of Q, s and y being known in terms of x from the known length of CB, and of the string; and the velocity of P is then found from the preceding equation.

The problem may be solved otherwise as follows:

Having found, as before, that the motive force of P is

$$\frac{du}{dt} \cdot \frac{P}{g} \cdot \frac{u}{v},$$

we have, for the determination of the mass M, which, when accelerated with Q has the same motive force, or offers the same resistance to motion, the equation

$$M \frac{dv}{dt} = \frac{du}{dt} \cdot \frac{P}{g} \cdot \frac{u}{v} \therefore M = \frac{P}{g} \cdot \frac{u}{v} \cdot \frac{du}{dv};$$

hence the whole inertia of the system being $M + \dfrac{Q}{g}$, we have for the acceleration of Q

$$\frac{Q}{M + \dfrac{Q}{g}} = \frac{Q\,g\,v}{Q\,v + P\,u\,\dfrac{du}{dv}} = v\,\frac{dv}{dy} \text{ (page 123.)}$$

Consequently $Q\,g\,dy = Q\,v\,dv + P\,u\,du$, as before.

PROBLEM IX.—A perfectly flexible chain is wound round a cylinder, supported with its axis parallel to the horizon. Then the weight and dimensions of the cylinder being given, as also the weight and length of the chain: it is required to determine the time in which the chain, impelled by the force of gravity, will unwind itself; a given length being unwound at the commencement of the motion.

The moving force here always acting is due to that part of the chain which hangs down, and the resistance to be overcome is the rotatory inertia of the cylinder, and of the mass of chain which en-

velopes it. As, (page 189,) the square of the radius of gyration of the cylinder is $\frac{1}{2}R^2$, and that of a mere circumference of the same radius R, R^2, it follows that the system would offer the same resistance to rotation, if for the cylinder we substitute an indefinitely slender ring of the same radius, containing half the mass of the cylinder, and the whole mass of the unwound chain.

Let then 2W denote the whole weight of the cylinder, and w that of the chain; let the length of the chain be l, a the length hanging down at the commencement of the motion, and x the length hanging down at any time t''. Then the weight of any part being as its length, we have $l : x :: w : \frac{wx}{l}$, which expresses the moving force acting at the extremity of the radius at any time t''; the value of this force so acting to turn the system is $R\frac{wx}{l}$, and this, divided by the moment of inertia, will give the angular acceleration (158), and consequently the acceleration of the extremity of the radius, or of the descending chain, is

$$R^2\frac{wx}{l} \div R^2\left(\frac{W+w}{g}\right), \text{ that is, } \frac{d^2x}{dt^2}=g\frac{wx}{l(W+w)}.$$

Multiplying this by $2\frac{dx}{dt}$, we have

$$\frac{2\,d^2x}{dt^2}\cdot\frac{dx}{dt}=2g\frac{wx}{l(W+w)}\frac{dx}{dt}.$$

This, multiplied by dt and integrated, gives

$$\frac{dx^2}{dt^2}=v^2=g\frac{wx^2}{l(W+w)}+C.$$

To determine C we have the condition $v=0$ when $x=a$,

$$\therefore\ 0=g\frac{w\,a^2}{l(W+w)}+C;$$

hence, subtracting this equation from the former,

$$v^2=g\frac{w(x^2-a^2)}{l(W+w)}\therefore v=\frac{dx}{dt}=\sqrt{\left\{g\frac{w(x^2-a^2)}{l(W+w)}\right\}},$$

consequently

$$dt=\sqrt{\left\{\frac{l(W+w)}{gw}\right\}}\frac{dx}{\sqrt{\{x^2-a^2\}}},$$

the integral of which is (*Int. Calc.* art. 18),

$$t=\sqrt{\left\{\frac{l(W+w)}{gw}\right\}}\log.\frac{x+\sqrt{\{x^2-a^2\}}}{a},$$

which is the number of seconds occupied in unwinding the length $x-a$, the part a being unwound at the beginning; and when $x=l$

we have, for the number of seconds occupied in unwinding the whole chain,

$$t=\surd\{\frac{l(W+w)}{gw}\}\log.\frac{l+\surd\{l^2-a^2\}}{a}$$

PROBLEM X.—To a point in the circumference of the base of an upright cylinder, standing on a smooth horizontal plane, is fastened one extremity of a string, and to the other a weight P. Now a given velocity is communicated to P in a direction perpendicular to the string: to determine the circumstances of the motion (fig. 137).

Let A be the point to which the string is fastened, and B the point of contact at any time t''; let also AOB$=\omega$, BP$=z$, AO$=a$, tension of the string$=$T, then, taking ON, NP for the rectangular co-ordinates of P at the time t'', we have, since the acceleration of P in the direction PB is $\frac{T}{P}$, these equations of its motion in the directions of the co-ordinates, viz.

$$\frac{d^2x}{dt^2}=-\frac{T}{P}\sin.\omega,\ \frac{d^2y}{dt^2}=-\frac{T}{P}\cos.\omega \ldots\ldots (1),$$

the negative signs being used because the accelerative force $\frac{T}{P}$, being in the direction PB, tends to diminish the co-ordinates.

Multiplying the first of these equations by cos. ω, and the second by sin. ω, and subtracting, we have

$$\cos.\omega\frac{d^2x}{dt^2}-\sin.\omega\frac{d^2y}{dt^2}=0 \ldots\ldots (2).$$

Again, the angular velocity of the string, since OBN$'=\omega$, is $\frac{d\omega}{dt}$; and, therefore, the absolute velocity of P in direction PQ, of its path, is $z\frac{d\omega}{dt}$, and therefore its components, in the directions of the co-ordinates, are $z\cos.\omega.\frac{d\omega}{dt}$, and $-z\sin.\omega\frac{d\omega}{dt}$, consequently, by differentiating,

$$\frac{d^2x}{dt^2}=\frac{dz}{dt}\cos.\omega\frac{d\omega}{dt}-z\sin.\omega\frac{d\omega^2}{dt^2}+z\cos.\omega\frac{d^2\omega}{dt^2}.$$

$$\frac{d^2y}{dt^2}=-\frac{dz}{dt}\sin.\omega\frac{d\omega}{dt}-z\cos.\omega\frac{d\omega^2}{dt^2}-z\sin.\omega\frac{d^2\omega}{dt^2}.$$

Substituting these in the equation (2), there results

$$\frac{dz}{dt}.\frac{d\omega}{dt}+z\frac{d^2\omega}{dt^2}=0\ \therefore\ \frac{dz}{z}=-\frac{d^2\omega}{dt^2}\div\frac{d\omega}{dt},$$

and integrating each side, $\log.\frac{c}{z}=\log.\frac{d\omega}{dt}\therefore\frac{c}{z}=\frac{d\omega}{dt}$, the angular velocity of the string at any time t''. To determine c let v_1 be the initial velocity of P when $z=b$, then the initial angular velocity is $\frac{v_1}{b}=\frac{c}{b}\therefore c=v_1$; consequently, $\frac{v_1}{z}=$ angular velocity of the string, $v_1=$ absolute velocity of P; hence, P continually moves with its initial velocity. As the centrifugal force of P is that to which the tension of the string is due, we have, $\mathrm{T}=\mathrm{P}\frac{v_1^2}{z}=\frac{\mathrm{P}v_1^2}{l-a\omega}$, l being the whole length ABP of the string.

PROBLEM XI.—To determine the path of a projectile in a resisting medium.

Let R represent the resistance of the medium in opposition to the motion of the body; then the forces acting upon it, in the directions of its horizontal and vertical co-ordinates, are

$$\frac{d^2x}{dt^2}=-\mathrm{R}\frac{dx}{ds},\ \frac{d^2y}{dt^2}=-g-\mathrm{R}\frac{dy}{ds},$$

which are the equations of the motion; and by means of which, when the law of resistance is known, the nature of the trajectory may be determined. The law generally received is, that the resistance varies as the square of the velocity; assuming, therefore, this to be the case, the foregoing equations are

$$\frac{d^2x}{dt^2}=-mv^2\frac{dx}{ds},\ \frac{d^2y}{dt^2}=-g-mv^2\frac{dy}{ds},\text{ or, since }v^2=\frac{ds^2}{dt^2}$$

the equations are the same as

$$\frac{d^2x}{dt^2}=-m\frac{ds^2}{dt^2}.\frac{dx}{ds},\ \frac{d^2y}{dt^2}=-g-m\frac{ds^2}{dt^2}.\frac{dy}{ds}.$$

As the second members of each of these equations contain only first differential co-efficients, and as the values of these co-efficients remain the same, however we change the independent variable, (*Diff. Calc.* p. 99,) the equations may be written

$$\frac{d^2x}{dt^2}=-m\frac{ds}{dt}.\frac{dx}{dt},\ \frac{d^2y}{dt^2}=-g-m\frac{ds}{dt}.\frac{dy}{dt}\ .\ (1)\ \therefore\frac{\frac{d^2x}{dt}}{\frac{dx}{dt}}=-m\,ds$$

Integrating this equation we have

$$\log.\frac{dx}{dt}=-ms+\mathrm{C}\ \therefore\frac{dx}{dt}=c\,e^{-ms}\ \ldots\ (2),$$

e being the base of the hyperbolic system of logarithms, and c the number whose logarithm is C.

The determination of c will depend upon the initial conditions

of the motion; let the initial velocity be v_1, and α the angle it makes with the axis of x. The component of v_1, according to this axis, is $v_1 \cos.\alpha$, and this same component is what $\frac{dx}{dt}$ becomes when $s=0$, that is to say $v \cos.\alpha = c\,e^0 = c$; hence, substituting this value of c in the preceding equation, we have $\frac{dx}{dt} = v_1 \cos.\alpha\, e^{-ms} \ldots (3)$.

Again: multiplying the first of equations (1) by $\frac{dy}{dt}$, the second by $\frac{dx}{dt}$, and taking their difference, we have

$$\frac{d^2x}{dt^2}\cdot\frac{dy}{dt} - \frac{d^2y}{dt^2}\cdot\frac{dx}{dt} = g\frac{dx}{dt};$$

dividing each side by $\frac{dx}{dt}$ the result will be $-\frac{dx}{dt}\cdot\frac{d\cdot\frac{dy}{dx}}{dt} = g$;

or, putting y' for $\frac{dy}{dx}$, $-\frac{dx}{dt}\cdot\frac{dy'}{dt} = g$, an equation containing only first differential co-efficients, and which we may, therefore, write thus (*Diff. Calc.* p. 99)

$$-(dx)(dy') = g\,(dt^2);$$

the parenthesis intimating that the independent variable is arbitrary. Also, from equation (3), $(dt^2) = \frac{(dx^2)}{v_1^2 \cos.\alpha\, e^{-2ms}}$; hence, by substitution, $-v_1^2 \cos.^2\alpha\, e^{-2ms}\frac{(dy')}{(dx)} = g \ldots (4)$.

Now $\frac{(ds)}{(dx)} = \sqrt{1+(y'^2)}$ $\therefore$ $(dx) = \frac{(ds)}{\sqrt{1+(y'^2)}}$, substituting this value of (dx) in (4) and omitting the parentheses, we shall have the equation $\sqrt{1+y'^2}\,.\,dy' = -\frac{g\,e^{2ms}\,ds}{v_1^2 \cos.^2\alpha}$,

of which the integral is

$$y'\sqrt{1+y'^2} + \log.(y' + \sqrt{1+y'^2}) = C - \frac{g\,e^{2ms}}{m\,v_1^2 \cos.^2\alpha} \ldots (5).$$

In order to determine C we must revert to the value of y' or $\frac{dy}{dx}$, at the commencement of the motion; this value is $y' = \tan.\alpha$, corresponding to which we also have $x = 0$, $y = 0$, $s = 0$;

$$\text{hence } C = \left\{ \begin{array}{c} \tan.\,\alpha \sqrt{1+\tan.^2\alpha} + \\ \log.\,(\tan.\,\alpha + \sqrt{1+\tan.^2\alpha}) + \dfrac{g}{m\,v_1^2 \cos.^2\alpha} \end{array} \right\}$$

therefore the value of C in the foregoing equation may be regarded as known.

To determine a differential equation between x and y we may eliminate e^{2ms} by means of equation (4) and (5); we shall thus have

$$dx = \frac{dy'}{m\,\{y'\sqrt{1+y'^2} + \log.\,(y' + \sqrt{1+y'^2}) - C\}};$$

also, since $dy = y'dx$,

$$dy = \frac{ydy'}{m\,\{y'\sqrt{1+y'^2} + \log.\,(y' + \sqrt{1+y'^2}) - C\}}.$$

These equations are too complicated to admit of integration in finite terms, otherwise we might now obtain the values of x and y in functions of y', and then, eliminating y', we should have a single equation in x and y which would be the equation of the path sought. As it is, we can only obtain an approximation to the form of the curve described, which approximation may, however, be carried to any degree of exactness. For the method of effecting the actual construction of the trajectary the student may refer to Mr. Barlow's Mechanics, in the Encyclopædia Metropolitana, or to Venturoli's Mechanics, and for the general theory of motion in a resisting medium, he may consult the second book of Mr. Whewell's Dynamics.

We shall here terminate this miscellaneous collection of problems, and must refer the student, for a more extensive variety, to the Ladies' and Gentlemen's Diaries, and to Leybourn's Mathematical Repository, works which cannot be too strongly recommended to the attention of the mathematical student, and to which our obligations are due for several of the foregoing examples.

NOTE.

Page 21.

Poisson's Proof of the Parallelogram of Forces.

(180.) When two equal forces act on a point according to different directions, their resultant, whatever it be in intensity, must necessarily bisect the angle between these directions, as shown at art. (7); and to determine the intensity of this resultant, M. Poisson proceeds as follows:

Let mA, mB (fig. 138,) be the directions of the components, whose common value call P; also let $2x$ represent the angle AmB, then mC being the direction of the resultant, we shall have $AmC = BmC = x$. The intensity of this resultant can depend only on the quantities P and x, of which, therefore, it is some unknown function. Representing, then, by R the value of the resultant, we shall have $R = f(P, x)$. In this equation R and P are the only quantities of which the numerical value varies with the unit of force that may have been chosen; their ratio $\frac{R}{P}$ is independent of this unit; whence we may conclude that it must be simply a function of x, and consequently that the function $f(P, x)$ is of the form $P . \phi x$. We therefore have

$$R = P . \phi x,$$

aud the question is reduced to the determination of the form of the function ϕx.

In order to this, let us draw arbitrarily through the point m, the four lines mA', mA'', mB', mB''; suppose the four angles $A'mA$, $A''mA$, $B'mB$, $B''mB$, equal among themselves, and represent each of them by z; this done, decompose the force P, directed according to mA, into two equal forces, directed according to mA' and mA'', that is to say, regard this force P as the resultant of two equal forces whose value is unknown, and which acts in the given directions mA', mA''. Representing the common value of these components by Q, we shall have

$$P = Q . \phi z,$$

for there ought to exist among the quantities P, Q, and z, the same relation as among the quantities R, P, and x. Decompose now the same force P, acting in the direction mB, into two forces Q, acting

in the directions mB' and mB''; the two forces P are thus replaced by the four forces Q; the resultant of these must, therefore, coincide in magnitude and direction with the force R, which is the resultant of the two forces P. Now, calling Q' the resultant of the two forces Q, acting in the directions mA' and mB', and observing that $A'mC=B'mC=x-z$, this force Q' will take the direction mC, and we shall have

$$Q'=Q\,.\,\phi\,(x-z).$$

In like manner, the resultant of the two other forces Q, which act in the directions mA'' and mB'', will take the direction mC, since this line divides the angle $A''mB''$ into two equal parts, and because $A''mC=B''mC=x+z$, we shall have

$$Q''=Q\,.\,\phi\,(x+z),$$

Q'' representing this resultant. The two forces Q' and Q'' being directed according to the same line mC, their resultant, which is also that of the four forces Q, will be equal to their sum; we must therefore have

$$R=Q'+Q''$$

but we already have $R=P\cdot\phi x=Q\cdot\phi z\cdot\phi x$; substituting then this value of R and those of Q' and Q'' above, and then suppressing the common factor Q, there results

$$\phi x\cdot\phi z=\phi\,(x-z)+\phi(x+z).$$

This is the equation which we must now solve in order to obtain the value of ϕx, or, which amounts to the same thing, that of ϕz. This is effected in a very simple manner by the following considerations.

Let us develope $\phi\,(x-z)$ and $\phi\,(x+z)$, according to the powers of z, by means of Taylor's theorem; let us substitute these two series in our equation, and then divide all its terms by ϕ, we shall thus have

$$\phi\,z=2\,\{1+\frac{d^2\phi x}{\phi x\,dx^2}\cdot\frac{z^2}{2}+\frac{d^4\phi x}{\phi x\,dx^4}\cdot\frac{z^4}{2\cdot3\cdot4}+\&c.\}$$

Now as ϕz ought not to contain x, x cannot enter into the coefficients

$$\frac{d^2\,\phi x}{\phi x\,dx^2},\ \frac{d^4\,\phi x}{\phi x\,dx^4},\ \&c.$$

all these quantities, therefore, must be constants, that is, independent of the variables x and z. Let b be the value of the first, we have

$$\frac{d^2\,\phi x}{dx^2}=b\phi x;$$

whence, by successive differentiation, we have

$$\frac{d^4\,\phi x}{dx^4}=b\,\frac{d^2\,\phi x}{dx^2}=b^2\,\phi x,\frac{d^4\,\phi x}{\phi x\,dx^4}=b^2$$

$$\frac{d^6\,\phi x}{dx^6}=b^2\frac{d^4\,\phi x}{dx^4}=b^3\,\phi x,\ \frac{d^6\,\phi x}{\phi x\,dx^6}=b^3$$
$$\text{\&c.}\qquad\text{\&c.}\qquad\qquad\text{\&c.}$$

and, consequently,

$$\phi z=2\left\{1+\frac{bz^2}{2}+\frac{b^2\,z^4}{2\cdot 3\cdot 4}+\frac{b^3\,z^6}{2\cdot 3\cdot 4\,.\,5\cdot 6}+\text{\&c.}\right\}.$$

or else in replacing b by another constant $-a^2$ which is allowable,

$$\phi z=2\left\{1-\frac{a^2\,z^2}{2}+\frac{a^4\,z^4}{2\cdot 3\cdot 4}-\frac{a^6\,z^6}{2\cdot 3\cdot 4\cdot 5\,.\,6}+\text{\&c.}\right\}.$$

We recognise the series within the parenthesis to be the developement of cos. az; then $\phi z=2$ cos. az, and putting x in place of z

$$\phi x=2\ \text{cos.}\ ax$$
$$\therefore\ R=2\ P\ \text{cos.}\ ax.$$

To determine the quantity a, which we know to be independent of x, we may observe that when $x=90°$ the two forces P are directly opposite; their resultant R is then 0; so that we must have

$$\text{cos.}\ (a\cdot 90°)=0;$$

which requires that a be an uneven whole number. This whole number must be 1; for if we had $a>1$, for example $a=3$, the resultant R would become 0 for $x=\frac{90°}{3}$ the two forces P would then equilibrate without being opposite, which is impossible. We shall have, therefore,

$$R=2\ P\cdot\text{cos.}\ x.$$

This result establishes the property otherwise deduced in art. 11, viz. that any two equal forces have for their resultant the diagonal of the rhombus constructed on the straight lines which represent them in magnitude and direction, and the property may now be generalized as in the text; the equation above deduced being the only thing connected with the general theorem of the parallelogram of forces of difficulty to establish.


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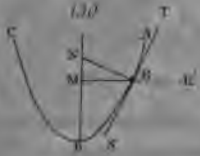
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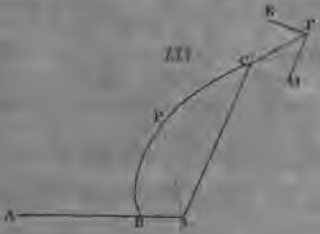
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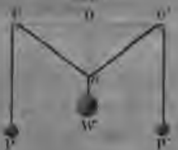
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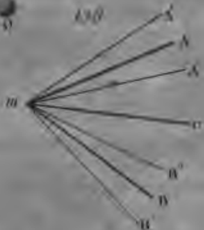
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SOLUTIONS AND NOTES BY THE EDITOR.

PROBLEM XI.—Page 92.—It will here be convenient to use fig. 49, (plate p. 90,) which is used in the solution of prob. III, p. 85, observing that in the present question AC=CB and ∴ that the perpendicular CE falls upon the middle of AB and bisects the angle ACB, which is here denoted by θ. Then, as in the problem cited, we shall have $P \,.\, CE = W \cdot \frac{CF}{2}$, (1), by using W instead of his 2P; but $CE = CB \cdot \cos.\frac{\theta}{2}$, and $CF = CB \,.\, \cos.\,\theta$, ∴ by substitution and reduction (1) becomes $2\,P \cos.\frac{\theta}{2} = W \cos.\,\theta$, (2); but (see *Young's Trigonometry*, p. 37,)* we have $\cos.\,\theta = 2\cos.^2\frac{\theta}{2} - 1$ ∴ by subst. and reduc. (2) becomes $\cos.^2\frac{\theta}{2} - \frac{P}{W}\cos.\frac{\theta}{2} = \frac{1}{2}$, hence by quadratics we have $\cos.\frac{\theta}{2} = \frac{P}{2W} \pm (\frac{P^2}{4W^2} + \frac{1}{2})^{\frac{1}{2}}$ as required.†

NOTE.—It may be well to observe that the sign + is to be used before the radical in the value of $\cos.\frac{\theta}{2}$ when the rod lies as in the figure; but the sign — is to be used when it lies in an opposite direction.

PROBLEM XII.—Let us imagine the cone to be placed as in the problem. Let P denote the force sought, and p the perpendicular from the edge on which the cone stands to the line of its direction; regarding the edge as a fixed point round which the cone can turn freely in a vertical plane passing through its axes. Now by putting 284=W, and p'= the perpendicular from the edge to a vertical line passing through the centre of gravity of the cone, we shall have as in the last problem, $P \,.\, p = W \,.\, p'$, (1); hence since $P \,.\, p$=const. and that P is to be a minimum, p must be a maximum: but it is easy to see that p is a maximum when it coincides with the lower side of the cone

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* American edition.

† The answers given by Mr. Young to problems XI. XII. and XIII. are incorrect.

$$\therefore p=\sqrt{\{20^2+3^2\}}=\sqrt{\{400+9\}}=\sqrt{\{409\}},$$

it is also easy to find $p'=0\cdot196+2$; hence, by restoring the value of W and substituting the values of p, p', (1), gives

$$P=\frac{0\cdot196\times142}{\sqrt{409}}=1\cdot377 \text{ cwt.}$$

acting perpendicular to the lower side of the cone.

NOTE.—Since a vertical line through the edge passes between the centre of gravity and vertex of the cone, P acts towards the horizon; in the contrary case it must act in an opposite direction.

PROBLEM XIII.—See fig. 58, p. 98. Let ABC denote the segment sought, having G for its centre of gravity and GP for a vertical line passing through it. Then by what was shown at p. 72 when the segment is at rest, GP must pass through the point of contact P of the segment with the horizontal plane RP. Now put a=half the transverse axis, and $AX=x$, then by prob. X, p. 68,

$$AG=\frac{8ax-3x^2}{12a-4x},$$

but since P is to coincide with A (per prob.) we have $AG=GP$, $\therefore$ the normal $GP=\dfrac{8ax-3x^2}{12a-4x}$; but see *Young's Analytical Geom.* (edition by Williams, just published by Carey, Lea & Blanchard, Philadelphia,) p. 135, we have $GP\,(=AG)=\frac{1}{4}a$. Hence, by comparing the values of GP we have $a=\dfrac{8ax-3x^2}{3a-x}$, or by reduction

$x^2-3ax=-a^2$, $\therefore$ by quadratics $x=\dfrac{a}{2}(3-\sqrt{5})$ as required.

PROBLEM XIV.—Let normals to the wall and inclined plane be drawn through the extremities of the beam; then when the beam is at rest they will evidently intersect each other in a vertical line which passes through the centre of gravity of the beam, for the resultant of $-P$, $-T$, which denote the reactions of the wall and plane must equal W and act directly opposite to it. Let the line of common section of the plane of the normals and horizon be taken for the axis of x. and a vertical line through the extremity of the beam which is on the inclined plane be taken for the axis of y; now it is evident that the reactions of the plane and wall act in the directions of their respective normals, $\therefore$ put z=the angle made by the normal to the inclined plane with the horizon, and we shall have $-T\cos. z+P=0$, (1); $-T\sin. z+W=0$, (2) for the resultant of the forces which act on the beam when resolved in the directionsof the axes of x and y respectively, supposing the beam to

be kept at rest by the forces which affect it, (see art. 19, p. 24.) Again, since the beam is uniform, its centre of gravity is at its middle point, and hence, we shall evidently have, tan. $z=2$ tan. i, (3); by (1) and (2) we have, tan. $z=\frac{W}{P}$, $T=\sqrt{\{W^2+P^2\}}$;

hence, and by (3), $P=\frac{W}{2\tan. i}$; $T=\frac{\sqrt{(\sin.^2 i+\frac{1}{4}\cos.^2 i)}}{\sin. i}.W$ as required.

PROBLEM XV.—Let l denote the length of the ladder, $d=$ the distance which the man has ascended on it, and $d'=$ the distance of the common centre of gravity of the ladder and man from the lower end of the ladder measured on it, then by the nature of the centre of gravity (art 40, p. 60,) we have

$$d'=\frac{\frac{1}{2}\,l\,w+dW}{w+W},\ (1);$$

now (art. 36, p. 50,) the resultant of w and $W=w+W$, and it passes through their common centre of gravity, (or at the distance d' from the lower end of the ladder,) and is vertical to the horizon. Let a straight line be drawn from the lower end of the ladder to the point of intersection of a normal to the wall at the upper end of the ladder, and the vertical through the common centre of gravity of w and W; then imagine a plane to be drawn at right angles to the line (thus drawn) through the lower end of the ladder; hence we may consider the ladder as resting on the wall and plane, like the beam in the last problem; hence, by using the same notation and proceeding in the same manner as in the last problem, we have,

$$\tan. z=\frac{W'}{P},\ (2);\ T=\sqrt{(W'^2+P^2)}.\ (3).$$

By putting $w+W=W'$; we also evidently have

$$\tan. z=\frac{l}{d'}\times\tan. i,\ (4),$$

hence by (2) and (4) we have

$$P=\frac{d'\,W'}{l\,\tan. i},\ (5);\ T=\frac{\sqrt{(\sin.^2 i+\frac{d'^2}{l^2}\cos.^2 i)}}{\sin. i}.W',\ (6)$$

as required; where P=the pressure against the wall, and T= the thrust at the bottom of the ladder.

Solution of Question VII. proposed at page 211.

Let i, r_1, r_2, g, w, wk^2, denote the same things as in the author's solution; then the accelerating force of gravity down the plane $=g\sin i$. Let us suppose that the plane is perfectly smooth, and that the wheel is acted upon by the motive force wF at the point of contact P′ towards D; put $t=$the time from the origin of the motion, $x=$the space described by the centre down the plane, and $\theta=$ the angle described by the wheel around its axis. Then by (p. 202) the centre moves in the same manner that it would do if the forces were immediately applied to it without changing their direction $\therefore g\sin.i-\mathrm{F}=\frac{d^2x}{dt^2}$, (1), also the wheel turns in the same manner that it would do if the centre were absolutely fixed, but the force impressed to turn the wheel is wF whose effect evidently$=w\mathrm{F}r_2$, and the effect produced $=\frac{d^2\theta}{dt^2}\mathrm{S}r^2dm=\frac{d^2\theta}{dt^2}wk^2$, ($r=$the distance of any element of the wheel from its centre); hence by (art. 154, p. 184,)

$$r_2\mathrm{F}=k^2\frac{d^2\theta}{dt^2}, (2) \therefore \text{by (1) } r_2g\sin.i=r_2\frac{d^2x}{dt^2}+k^2\frac{d^2\theta}{dt^2}\ldots\ldots (3).$$

Multiply (3) by r_2, then if we suppose $r_2\theta=nx$, (4), (where $n=$a constant number), we shall have $r_2\frac{d^2\theta}{dt^2}=n\frac{d^2x}{dt^2}$, and (3) will give $\frac{d^2x}{dt^2}=\frac{r_2^2g\sin.i}{r_2^2+nk^2}$, $\therefore\frac{dx}{dt}=\frac{r_2^2g\sin.i}{r_2^2+nk^2}t$, (5), $x=\frac{r_2^2g\sin.i}{r_2^2+nk^2}\frac{t^2}{2}$; (6), where x, θ, t, are supposed to commence together, whence all the circumstances of the motion become known. If $n=1$, the wheel rolls without sliding, the force wF performing the office of the friction supposed by the author in the case which he has considered. If n is >1, then $r_2\frac{d\theta}{dt}=n\frac{dx}{dt}=$ the velocity of rotation is $>\frac{dx}{dt}=$ the velocity of the centre $=$ the velocity of translation, yet the wheel does not move up the plane as the author says it will, near the end of his solution; hence his remarks are incorrect.

Solution of Question IV. proposed at page 238.

Suppose (with the author), that B moves with the given velocity v towards D, then the space which it will describe in the time t (from the origin of the motion), $= vt$; put $\theta =$ the angle ABC, ($\theta' =$ its value at the origin), $r =$ the distance of any element dm, of the rod from the end B, M = the mass of the rod, R = BG, $g = 32.2$ft. (= the accelerating force of gravity), $dt =$ the constant element of the time, $x =$ the distance of dm (estimated on CD), from the initial position of B, then evidently $x = vt - r\cos.\theta$, (1). Now $g\cos.\theta dt\, dm =$ the force impressed on dm, and $-\dfrac{rd^2\theta}{dt} =$ the force received by it (in the instant dt) in a direction perpendicular to the rod, (supposing the forces tend to diminish θ); $\therefore$ the motion lost by dm, $= dm\left(\dfrac{rd^2\theta}{dt} + g\cos.\theta dt\right)$ and the momentary effect of this force to turn the rod about B (by the principle of the lever), $= dm\, r\left(r\dfrac{d^2\theta}{dt} + g\cos.\theta dt\right)$, hence by art. (154) we shall have by using S as the sign of integration relative to the mass of the rod,

$$\mathrm{S}dm\, r\left(r\frac{d^2\theta}{dt} + g\cos.\theta dt\right) = 0, \text{ or } \frac{d^2\theta}{dt^2}\mathrm{S}r^2dm + g\cos.\theta\mathrm{S}rdm = 0,$$

but by art. (42), (155) $\mathrm{S}rdm = \mathrm{RM}$, $\mathrm{S}r^2dm = \dfrac{4\mathrm{R}^2}{3}\mathrm{M}$, hence by substitution and reduction, we have $\dfrac{d^2\theta}{dt^2} + \dfrac{3g}{4\mathrm{R}}\cos.\theta = 0$, (2), multiply (2) by $d\theta$ and take the integral and we have $\dfrac{d\theta^2}{dt^2} + \dfrac{3g}{2\mathrm{R}}\sin.\theta = c =$ contrast. (3).

Suppose that at the origin the velocity V was impressed on the centre of gravity G of the rod in the direction GD′, then by changing r in (1) into R we have $x = vt - \mathrm{R}\cos.\theta$ (4) for the value of x which corresponds to the centre of gravity, hence evidently $\dfrac{dx}{dt} = \mathrm{V} = v + \mathrm{R}\sin.\theta'\dfrac{d\theta'}{dt}$ (at the origin), which gives $\dfrac{d\theta'^2}{dt^2} = \left(\dfrac{\mathrm{V}-v}{\mathrm{R}\sin.\theta'}\right)^2$ but (3) at the origin gives

$$\frac{d\theta'^2}{dt^2} + \frac{3g}{2\mathrm{R}}\sin.\theta' = c, \text{ hence } \left(\frac{\mathrm{V}-v}{\mathrm{R}\sin.\theta'}\right)^2 + \frac{3g}{2\mathrm{R}}\sin.\theta' = c,$$

$$\text{hence, and by (3) } dt = \frac{\pm d\theta}{\sqrt{\left(\frac{\mathrm{V}-v}{\mathrm{R}\sin.\theta'}\right)^2 + \frac{3g}{2\mathrm{R}}(\sin.\theta' - \sin.\theta)}} \quad (5);$$

the integral of (5) will give t in a function of θ and known quantities, $\therefore$ reciprocally θ will be found in a function of t and known quantities, hence x as given by (4) will be exhibited in terms of t and known quantities, $\because$ we have the value of x which corresponds to the position of the centre of gravity at any time, (t) and by putting $r = o$ in (1) we have $x = vt$ which gives the position of B at the same time, and the position of the rod is determined in all respects; and it may be observed that the sign $+$ is to be used in (5) before $d\theta$ when V is greater than v, but the sign $-$ in the contrary case.

Again, if the second term of the quantity under the radical in (5) is much smaller than the first, the integral may be readily found in a rapidly converging series; but if the second term is infinitely small relative to the first term, it may be neglected, and we shall have (when V is $> v$), $\frac{V - v}{R \sin.\theta'} dt = d\theta$ and by integration, and correction $\theta = \theta' + \frac{V - v}{R \sin.\theta'} t$, which will also give the value of θ when v is $>$ V these are the results which the author should have found in the case (of the general problem), which he appears to have considered; and it is easy to see that every thing else required in this case is found.

Note.—It is easy to see by (1) that the point B moves always with the velocity v; for it gives $\frac{dx}{dt} = v + r \sin\theta \frac{d\theta}{dt}$, but at the point B, $r = 0$, hence $\frac{dx}{dt} =$ the velocity of B $= v$, as it ought to do.

THE END.

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